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2019/26 August 2019
ISSN(ONLINE) 2284-0400

AGGREGATE PRODUCTIVITY GROWTH IN THE PRESENCE OF (PERSISTENTLY) HETEROGENEOUS FIRMS*

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Abstract

In this article we propose a new methodology for computing the aggregate productivity of an industry, its variations and decompositions of the latter into changes of individual productivities (within effect) and changes in industry composition (between effect). Current aggregate measures rely on some weighted average of individual productivities, and decompositions distinguish between the effect of productivities and weights on variations of the average. However such aggregate measure is incoherent with the disaggregate one (the two are computed with different methodologies), is subject to aggregation biases, arbitrariness in the choice of weights, and information loss. Such problems are particularly serious when heterogeneity among firms is high. We propose instead a geometric approach where aggregate productivity can be measured directly on industry data, but nevertheless its variations can be decomposed into between and within effects plus an heterogeneity effect. We show that our measure does not incur in many of the problems of the weighted average and we also present an empirical application to European data.

JEL codes: D24; C67; C81; O30

Keywords: Productivity measurement; Decomposition of aggregate productivity growth; Firm heterogeneity

*We benefited from participants' comments on this work (including its early version) at Comparative Analysis of Enterprise Data (CAED) Conference 2017, International Schumpeter Society (ISS) Conference 2018, Productivity Research Network (PRN) Conference 2018, Japan Association for Evolutionary Economics (JAFEE) conference 2019 and seminars at Jorge Tadeo Lozano University, Meijing University, the University of Tokyo and Chuo University. Le Li is a JSPS International Research Fellow (P18759) supported by the Japan Society for the Promotion of Science. Le Li also gratefully acknowledges the research support by the IBIMET-CNR (grant CrisisLab-ProCoPe) in the early stage of this article. This work has been partly supported by the European Commission under the H2020, GROWINPRO, Grant Agreement 822781.

1 Introduction

The considerable increase of the availability and reliability of firm-level data in recent years has given a much more solid empirical basis on at least two very important research lines in industrial economics focusing, respectively, on productivity and heterogeneity. The former has produced a progressive refinement of establishment- firm- industry- and country-level productivity and productivity change measures which have also fed the debate on whether we have entered a time of slow productivity growth in spite of the apparent wave of technological change (3rd and 4th industrial revolutions). This is not the place to discuss whether any slowdown has happened and, if it has, its causes. Rather, let us just notice that persistent heterogeneity is a very robust stylized fact, irrespectively of the level of statistical observation, in terms of e.g. productivity, profitability and other measures of firm characteristics and performance.

In turn, heterogeneity makes it more problematic to produce good aggregate measures of such characteristics and performance indicators and, most notably, of productivity. If firms had very similar individual productivities, computing aggregate values would be straightforward. Or, at least, if competition forces were pushing individual productivities towards those of the most efficient firms, homogeneity would be increasing and aggregation would become less problematic in the long run.

However this is not what happens in the real world. Firms operating in the same industry display large asymmetries in productivity, relative input intensities (also with the same relative prices) and high intertemporal persistence of such asymmetries (e.g. Baily et al., 1992; Baldwin and Rafiquzzaman, 1995; Bartelsman and Doms, 2000; Disney et al., 2003; Dosi, 2007; Syverson, 2011). Finally, such heterogeneity is persistent also when defining industries more and more narrowly, as summarized by Griliches and Mairesse (1999) " *We [...] thought that one could reduce heterogeneity by going down from general mixtures as "total manufacturing" to something more coherent, such as "petroleum refining" or "the manufacture of cement". But something like Mandelbrot's fractal phenomenon seems to be at work here also: the observed variability-heterogeneity does not really decline as we cut our data finer and finer. There is a sense in which different bakeries are just as much different from each others as the steel industry is from the machinery industry.*"

It is fundamental to note that firms are not heterogeneous in terms of a scalar which multiplies a commonly shared production function. In a deeper way, firms, and even plants of the same firm, differ substantially in terms of relative combinations of inputs and input/output relations.

Current measures of aggregate productivity (AP henceforth) for an industry are typically averages of the productivities of individual firms, weighted by some indicator of the shares of individual firms in the total size of the industry, i.e. either input (typically labour) or output (total sales) shares. However, weighted average measures have clear drawbacks. The choice of weights is somewhat arbitrary and different weights can produce very different results. Moreover there is a kind of incoherence when measuring individual productivities with one method (e.g. using a production function) and the aggregate productivity with a different method (a weighted average). This incoherence makes aggregate measure sensitive also to the level of observation of individual units: for instance computing AP as weighted average over productivities of establishments or productivities of firms yields, in general, very different results, though the latter are simply aggregations of the former. It seems therefore more natural to compute AP directly on aggregate data. Industry productivity could be simply computed using industry input output data and treating the industry as a single entity.

On the other hand, weighted average measures have apparently a very important advantage over such a "direct" measure. Usually we are not so much interested in measuring productivity *per se* but its variations, that is the "aggregate productivity growth" (APG henceforth), and, especially, to understand what caused such variations: changes of individual productivities or changes in the composition of the industry? Weighted average measures provide a straightforward answer to these question, since it is in principle easy to decompose the variation of the weighted average into a component due to the variation of firm productivities (the so-called "within" effect) and another component due to the variation of weights/shares (the "between" effect). Moreover, also the contribution of firms entering and exiting the market can be easily accounted for. On the contrary, it is usually believed that similar decompositions into within, between, entry and exit effects are not possible with direct measures.

In this article we challenge this view and propose a "direct" (i.e. using aggregate input and output data rather than averages of firm data) measure of AP which allows for a rigorous decomposition of its variations into within, between, and net entry effects. Moreover, we can further decompose the between effect into two parts, one which is a proper "shares" effect which captures the changes of the individual contributions to the size of the industry, and the other which instead captures the variations of the heterogeneity of the combinations of input used by productive units. Thus, our methodology is particularly appropriate when heterogeneity is significant and the standard decompositions would deliver a spurious measure of the between effect.

Our measure builds upon a representation of an industry empirical production activities set first proposed by Hildenbrand (1981) and further developed by Dosi et al. (2016) and it is based on some geometric properties of such a set. In addition to overcoming most of the problems encountered by weighted average measures, our methodology has also an additional important advantage: it is based solely upon the empirical production set and does not require to fit a production function in order to estimate total factor productivity (TFP henceforth).

Finally, we show that our methodology can indeed be applied to real data and we provide as an illustration and empirical application to some selected industries in France, Italy and the UK. A comparison between the results of our methodology and a standard one on the same empirical data set highlights the differences.

The rest of the article is organized as follows. Section 2 briefly reviews the current methods that compute APG decompositions and discusses the problems they face. Section 3 develops our own suggested measure and compares it to the current one by means of some artificial toy examples. Some mathematical details are left to appendix A, whereas the extension to the case of multiple outputs is discussed in appendix B. Section 4 presents instead a test of our methodology against a standard one on real empirical data of some selected industries in France, Italy and the UK. Finally, section 5 concludes.

2 Decomposition of APG: A brief discussion of the state of the art

Many empirical studies[1] try and measure the effect of changes of firm-level productivities and changes of industry composition on APG.

In this section we do not want to review this large and growing body of literature, but simply to sketch the methodology they rely upon (leaving aside many variations and refinements) and discuss its limitations. In order to do so, we will mainly refer to three methodologies introduced, respectively, by Baily et al. (1992), Grilliches and Regev (1995), and Haltiwanger (1997), and to the critical review given by Foster et al. (2001).

In subsection 2.1, we summarize three most influential decomposition methods, and in section 2.2 we discuss the potential issues with these methods.

2.1 APG and its decomposition

Foster et al. (2001) compare the major APG decomposition methods. AP at time t, denoted by P^t, is defined as the weighted average of the productivities of individual firms:

$$P^t := \sum_{i \in I^t} w_i^t p_i^t \tag{1}$$

where I^t is the set of all firms in the industry at time t, w_i^t is the weight/share (e.g. output share) of firm i in this industry, and p_i^t is the productivity of firm i. APG is then defined as the difference of the aggregate productivities between two (consecutive) time periods.

A decomposition method for APG was introduced by Baily et al. (1992):

$$\sum_{i \in I^t} w_i^t p_i^t - \sum_{i \in I^{t-1}} w_i^{t-1} p_i^{t-1} = \sum_{i \in C} w_i^{t-1} \Delta p_i^t + \sum_{i \in C} p_i^t \Delta w_i^t + \sum_{i \in N} w_i^t p_i^t - \sum_{i \in X} w_i^{t-1} p_i^{t-1} \tag{2}$$

where C denotes continuing firms, i.e. those active in both periods, N and X denote the set of firms that have, respectively entered and exited the industry, and the operator Δ represents change from time $t-1$ to time t, i.e. $\Delta p_i^t = p_i^t - p_i^{t-1}$ and $\Delta w_i^t = w_i^t - w_i^{t-1}$. The first term, originally called "fixed shares", represents a within-firm component given by each firm's productivity change, weighted by the initial shares in the industry. The second term, originally called "share effect", represents a between-firm component that reflects changing shares, weighted by the productivities of the final period. The last two terms represent the contribution of the firms that, respectively, entered and exited the industry (see Melitz and Polanec (2015) for a detailed discussion of the entry and exit effects).

A second decomposition method was introduced by Grilliches and Regev (1995). It only differs from the previous one in the weights used to compute the "within" and the "between" effects. Rather than using either the initial or the final weights, this method employs their averages:

$$\sum_{i \in I^t} w_i^t p_i^t - \sum_{i \in I^{t-1}} w_i^{t-1} p_i^{t-1} = \sum_{i \in C} \bar{w}_i \Delta p_i^t + \sum_{i \in C} \bar{p}_i \Delta w_i^t + \sum_{i \in N} w_i^t p_i^t - \sum_{i \in X} w_i^{t-1} p_i^{t-1} \tag{3}$$

[1] Among the others, Baily et al. (1992); Olley and Pakes (1996); Bartelsman and Dhrymes (1998); Dwyer (1998); Haltiwanger (1997) use data from US, Aw et al. (1997) from Taiwan, Liu and Tybout (1996) from Chile and Colombia, Grilliches and Regev (1995) from Israel, and Dosi et al. (2015) from France, Germany, and the UK.

where the bar over a variable indicates the average $\bar{w}_i = \frac{w_i^t + w_i^{t-1}}{2}$ and $\bar{p}_i = \frac{p_i^t + p_i^{t-1}}{2}$. In this decomposition, the first term can be interpreted as a within effect which is measured by the sum of productivities weighted by the average (across time) shares. The second term represents a between effect where the changes in the shares are indexed by the average firm-level productivities. The last two terms represent the contribution of the firm which entered and exited the industry.[2]

Finally, Haltiwanger (1997) suggests a refinement of decomposition (2) where 1) the "share effect" actually captures between effect and covariance terms, and 2) the between and the net entry effects are weighted by the deviations of firm productivities from the initial industry index:

$$\begin{aligned}\sum_{i \in I^t} w_i^t p_i^t - \sum_{i \in I^{t-1}} w_i^{t-1} p_i^{t-1} = & \sum_{i \in C} w_i^{t-1} \Delta p_i^t + \sum_{i \in C} \left(p_i^{t-1} - P^{t-1}\right) \Delta w_i^t + \sum_{i \in C} \Delta p_i^t \Delta w_i^t \\ & + \sum_{i \in N} w_i^t \left(p_i^t - P^{t-1}\right) - \sum_{i \in X} w_i^{t-1} \left(p_i^{t-1} - P^{t-1}\right)\end{aligned} \tag{4}$$

where a continuing firm whose output increases and a new entry will contribute positively to the index only if their productivities are higher than the aggregate, whereas an exiting firm will contribute positively only if its productivity was lower than the aggregate. Without such a deviation term the between effect may be different from zero even when all individual productivities remain constant if the share of entering and exiting firms are different.

2.2 Some problems with the current decomposition methods

Defining AP as a weighted average of the productivities of individual firms leaves open the question of which weights should be used. Different weights will, in general, deliver different aggregate values and different decompositions.

As an illustration, consider the extreme case of a hypothetical industry composed of the two highly heterogeneous firms described in Table 1.

Table 1: A hypothetical industry with two firms and one input

Firm	Labour	Output	Labour Productivity
A	1	100	100
B	100	1	0.01
Aggregate (Industry)	101	101	1

To compute the AP of this industry we can use either input or output shares as weights. With the former we obtain an AP of 1, whereas with the latter we obtain a value very close to 100, i.e. the productivity of firm A.

Alternatively, instead of computing AP as a weighted average, we could compute it directly, by considering the industry as one large firm producing the industry total output with the industry total input. This measure is 1 in our example, and it is equal to the value obtained with inputs as weights because we are considering a case with only one input. With multiple inputs this equality does not generally hold.

[2]To be more precise, Grilliches and Regev (1995) treat all entering and exiting firms as one firm.

In fact, when we have multiple inputs we normally aggregate them into a synthetic measure called total factor productivity (TFP). But the calculation of TFP requires the assumption of a specific production function and the empirical estimation of its parameters. Consider for instance a hypothetical industry composed of three firms using capital and labour as in Table 2.

Table 2: A hypothetical industry with three firms and two inputs

	L	K	Q	TFP	L-based Weights	K-based Weights	Q-based Weights
Firm 1	30	10	20	0.877	0.882	0.625	0.357
Firm 2	1	3	6	4.559	0.029	0.188	0.107
Firm 3	3	3	30	10.000	0.088	0.188	0.536
	L	K	Q	TFP	w.a. TFP_i	w.a. TFP_i	w.a. TFP_i
Industry	34	16	56	1.989	1.786	3.285	6.161

* w.a. stands for weighted average.

The first three columns of Table 2 report input-output data for each firm, the fourth column gives the corresponding TFP values under the assumption that the production function is a Cobb-Douglas with constant returns to scale:

$$TFP_i = \frac{Q_i}{L_i^{0.75} K_i^{0.25}} \tag{5}$$

and the three rightmost columns give the weights computed, respectively, on labour, capital and output. Finally the bottom row gives the aggregate values for L, K and Q, the aggregate TFP directly computed on such values (treating the industry as one production unit with the same production function) and the last three columns give instead the TFP values computed as weighted averages using, respectively, L, K and Q shares as weights.

Also in this case APs greatly vary with the chosen weights and they also vary from AP/TFP measure directly computed on the aggregate input-output data rather than as a weighted average of individual TFPs.

This "direct" aggregate measure presents some important advantages over weighted averages. The first and most important feature is that productivity is computed exactly in the same way for the individual firm and for the aggregate, whereas standard measures are defined in radically different ways at firm (TFP) and aggregate (weighted average) levels. As a consequence, the direct measure is invariant to changes of the level of aggregation at which individual units are observed. Suppose for instance that some production units merge into a single entity (for instance two firms merge legally, if our unit of analysis are firms) or simply we change our level of observation from individual establishment to multi-establishment firms, keeping exactly the same input-output structure. Weighted average measures would change[3] although the input-output structure of the industry has

[3]To be more precise, AP will be different if computed using outputs as weights and using TFP with multiple inputs. It would not change in the case of a single input.

remained exactly the same. The productivity computed directly on the aggregate would instead remain unchanged. For instance, imagine that firms 2 and 3 of Table 2 merge or, equivalently, that 2 and 3 are establishments of the same firm and establishment data are no longer available forcing us to rely on firm-level data only. It is easy to show that all APs computed as weighted averages would change and, in particular, AP would decrease by 3% when computed with labour shares, increase by 10% with capital shares and decrease by 10% with output share, whereas TFP directly computed on industry input-output data would remain obviously unchanged, correctly signalling that nothing as changed in the industry.

More in general, no matter the method adopted, when we try to summarize multiple pieces of information, e.g. quantities of inputs and outputs, into an aggregate value, e.g. the productivity level, we inevitably lose information. Thus, since every time we compute a productivity value we lose information, in principle it is preferable to minimize the number of such computations. Now, if we compute AP as a weighted average of the productivities of n firms we will lose information $n+1$ times, whereas if we compute productivity directly on the aggregate level, we only do it once. More information is preserved when moving from firm to industry level if we aggregate firms' production activities rather than their productivities.

Finally, when we compute APG, we have to make an additional arbitrary choice, that is the weights of either the initial, or the final, or some intermediate time. Also in this case the choice will matter and different weights may produce significantly different decompositions.

To summarize, it seems more reasonable to define productivity for an industry exactly as we define it for a firm, rather than computing it as an average. But then the next question is whether without using the weighted-average we can still perform a decomposition of AP variations into within, between and net entry components. Contrary to what is usually believed, in the next session we show that the answer to this question is yes, provided we use an appropriate methodology.

3 Productivity growth in firms and industries: a unified framework

In this section we propose our decomposition method of APG. We start with an empirical representation of an industry as a set of heterogeneous firms first introduced by Hildenbrand (1981) and later developed by Dosi et al. (2016). Such a representation does not assume the existence of a production function, but nevertheless allows to compute rigorous aggregate measures of productivity and, as we show below, decompositions of their variation which preserve coherence when passing from individual firms to industry aggregates. We first introduce some notation and definitions in subsection 3.1. Then, in subsection 3.2, we propose our measure of productivity both for individual firms and for an industry and show how the latter can be decomposed into a "within" and a "between" effect. In section 3.3, we further discuss the role of heterogeneity among firms in APG. In section 3.4, we extend our proposed decomposition method by taking into account firm entry and exit and, finally, in section 3.5 we illustrate our methodology with some toy examples, before presenting, in section 4, an empirical application to real data.

3.1 Notation and definitions

Following Koopmans (1977) and Hildenbrand (1981) we represent the actual technique of a production unit (e.g. firm) i by means of the vector of its *production activity*:

$$a_i^t = \left(\alpha_{i,1}^t, \cdots, \alpha_{i,(l-1)}^t, \alpha_{i,l}^t\right) \in \mathbb{R}_+^l, \tag{6}$$

where $\alpha_{i,l}^t$ is the output in period t and $\left(\alpha_{i,1}^t, \cdots, \alpha_{i,(l-1)}^t\right)$ is the vector of inputs. In order to keep our formalism simpler, in this section we confine our discussion to activities producing only one output. However, in appendix B we will show that our methodology can be easily extended and deal with production activities with more than one output. If I^t denotes the set of all production units within one industry at time t, the aggregate (industry) production activity can be defined as:

$$d^t = \left(\beta_1^t, \cdots, \beta_{l-1}^t, \beta_l^t\right) = \left(\sum_{i\in I^t} \alpha_{i,1}^t, \cdots, \sum_{i\in I^t} \alpha_{i,(l-1)}^t, \sum_{i\in I^t} \alpha_{i,l}^t\right) \in \mathbb{R}_+^l \quad , \tag{7}$$

i.e. the sum of all individual firm production activities in the industry.

The productivity p_i^t of the production unit i at time t can be measured as the tangent of the angle $\theta(a_i^t)$ that the vector a_i^t forms with the space of inputs (Dosi et al., 2016). To give the intuition behind this measure of productivity, let us consider the case of only one input. Clearly, the larger the angle that the vector representing a production activity forms with the input axis, and therefore the smaller the angle it forms with the output axis, the more productive is the activity. By extension to the case of multiple inputs we obtain the following productivity indicator:

$$p_i^t := tg\left(\theta(a_i^t)\right) = \frac{\alpha_{i,l}^t}{||pr\left(a_i^t\right)||} \tag{8}$$

where the map

$$\begin{aligned} pr : \mathbb{R}^l &\to \mathbb{R}^{l-1} \\ (x_1, \cdots, x_{l-1}, x_l) &\mapsto (x_1, \cdots, x_{l-1}) \end{aligned}$$

is the projection map on the space of inputs[4].

Similarly, we define the AP of the industry at time t, denoted by P^t, as

$$P^t := tg\left(\theta(d^t)\right) = \frac{\beta_l^t}{||pr\left(d^t\right)||} \quad . \tag{9}$$

Notice that whereas $\beta_l^t = \sum_{i\in I^t} \alpha_{i,l}^t$ is the total output of the industry, in general $||pr\left(d^t\right)|| \neq \sum_{i\in I^t} ||pr\left(a_i^t\right)||$, the equality holding either in the case of a unique input or, in the case of multiple inputs, only when all the vectors $pr(a_i^t)$ lie on the same line and therefore production activities are perfectly homogeneous and differ only in their scale. If instead techniques are heterogeneous and firms use different combinations of inputs, the inequality $||pr\left(d^t\right)|| \neq \sum_{i\in I^t} ||pr\left(a_i^t\right)||$ holds. This heterogeneity component is an important feature of our model and we will further exploit it later in the article.

[4]This can be easily generalised to multi-output case simply considering a different projection map (see Dosi et al. (2016)).

3.2 Decomposing APG into *within* and *between* Effects

AP, defined in equation (9), can be further written as a weighted average of individual productivities p_i^t, since

$$P^t = \frac{\beta_l^t}{||pr\,(d^t)\,||} = \frac{\sum_{i\in I^t}\alpha_{i,l}^t}{||pr\,(d^t)\,||} = \sum_{i\in I^t}\frac{||pr\,(a_i^t)\,||}{||pr\,(d^t)\,||}\frac{\alpha_{i,l}^t}{||pr\,(a_i^t)\,||}$$

from which, by (8), we get the decomposition

$$P^t = \sum_{i\in I^t} w_i^t p_i^t \tag{10}$$

where the weights

$$w_i^t := \frac{||pr\,(a_i^t)\,||}{||pr\,(d^t)\,||} \tag{11}$$

represent the input-based weights defined as the relative length of individual input vectors $||pr\,(a_i^t)\,||$ over industry input vector $||pr\,(d^t)\,||$. As already mentioned, the length of the industry input vector $||pr\,(d^t)\,||$ is not necessarily equal to the sum of the lengths of the individual input vectors $||pr\,(a_i^t)\,||$, thus $\sum_{i\in I^t} w_i^t$ is not necessarily equal to 1.

Equality (10) indicates that AP can be written as a weighted average of individual productivities p_i^t. However, it is important to stress a fundamental methodological difference between how we obtain this weighted average and the standard approaches that we briefly surveyed in section 2 above. In the latter, AP is **defined** and hence computed as a weighted average of individual productivities, no matter how the latter are measured. On the contrary, in our framework AP is defined and computed in exactly the same way as we compute individual productivities, as it clearly emerges by comparing definitions (8) and (9), not as a weighted average. Equation (10) shows that our AP has the property of being equal to a weighted average, but it is neither defined nor computed in that way. Moreover, the measure of AP we propose is also a straightforward generalization of the one-input-one-output case. When there is only one input, the industry input vector degenerates to one number and the tangent of the angle we use to measure AP, according to (9), becomes the quotient of the sum of all outputs divided by the sum of all inputs. Input based weights in (11), $w_i^t = \frac{\alpha_{i,1}^t}{\sum_{i\in I^t}\alpha_{i,1}^t}$ are nothing else than input shares with sum equal to 1.

Now we show that our measure of AP can be decomposed and the standard effects outlined by the traditional literature can indeed be easily computed also in our framework. For the sake of simplicity we first introduce our decomposition method only for the set C of continuing firms, i.e. all those that are active both in both periods $t-1$ and t. Entry and exit will be introduced later in section 3.4.

Such continuing firms are described at time $t-1$ by the vector set $\{a_i^{t-1}\}_{i\in C} \in \mathbb{R}_+^l$ and at time t by the vector set $\{a_i^t\}_{i\in C} \in \mathbb{R}_+^l$. Let d^{t-1} and d^t, computed according to (7), represent the corresponding aggregates at time $t-1$ and t respectively. Given all the production activity vectors, aggregate and individual productivities at $t-1$ and t can be easily computed according to (9) and (8) respectively.

We can now decompose APG, defined as the difference of AP between two consecutive years, into within-firm and between-firm components[5]:

$$\Delta P^t = \underbrace{\sum_{i \in C} \bar{w}_i \Delta p_i^t}_{Within} + \underbrace{\sum_{i \in C} \bar{p}_i \Delta w_i^t}_{Between} , \qquad (12)$$

where Δ represents the variation from year $t-1$ to year t, whereas $\bar{w}_i$ and $\bar{p}_i$ are averages of weights and productivities respectively.

In the above decomposition, the *within* term represents the contribution given to APG by the variations of the individual productivities and it is therefore similar to the "within" effect in the current literature. The *between* term presents instead an important difference when compared to the "between" effect in the standard literature. In the latter, the weights w_i^t are defined either as input or output shares and, in both cases, $\sum_{i \in C} w_i^t = 1$. This is not the case in our decomposition (12) where, since the sum of the lengths of individual input vectors is not necessarily equal to the length of the sum of individual input vectors, i.e. the length of the industry input vector, we have, in general, $\sum_{i \in C} w_i^t \neq 1$. We will discuss this point in details in the next subsection and show that, actually, our *between* effect can be further decomposed.

3.3 Further Decomposing the *Between* Effect

In this section, we show that we can further decompose our *Between* term into two parts, an input weights component and a heterogeneity component.

Given the individual a_i^t and aggregate d^t production activities, consider their projections on the input space and call them $pr\left(a_i^t\right)$ and $pr\left(d^t\right)$. Figure 1 provides a graphical representation for the case with two inputs, capital K and labour L, and one output. Notice that if all individual production activities used the same proportion of inputs and differed only in scale and/or productivity, all the projection vectors $pr\left(d^t\right)$ and $pr\left(a_i^t\right)$ would overlap. On the other hand, the further away $pr\left(a_i^t\right)$ is from $pr\left(d^t\right)$, the more the combination of inputs used by firm i differs from the industry average combination. To measure this difference, we introduce φ_i^t which is the angle formed by the individual projection vectors $pr\left(a_i^t\right)$ and the industry projection vector $pr\left(d^t\right)$. Notice that given vectors d^t and a_i^t, and thus $pr\left(d^t\right)$ and $pr\left(a_i^t\right)$, it is easy to compute $\cos\varphi_i^t$ for each firm i at time t.

In the input space, we denote by b_i^t the projection of $pr\left(a_i^t\right)$ onto the industry projection vector $pr\left(d^t\right)$ (see Figure 1 for a graphical illustration). Notice that the length $||b_i^t||$ of the vector b_i^t can be regarded as the contribution of $pr\left(a_i^t\right)$ to the length of the industry input vector $pr\left(d^t\right)$. Hence, from now on, we will refer to the length $||pr\left(a_i^t\right)||$ of the firm input vector $pr\left(a_i^t\right)$ as the **actual input size** of firm i and to the length $||b_i^t||$ of the vector b_i^t as the **contributing input size** of firm i. It is easy to see that $||pr\left(d^t\right)|| = \sum_{i \in I^t} ||b_i^t||$.

Since

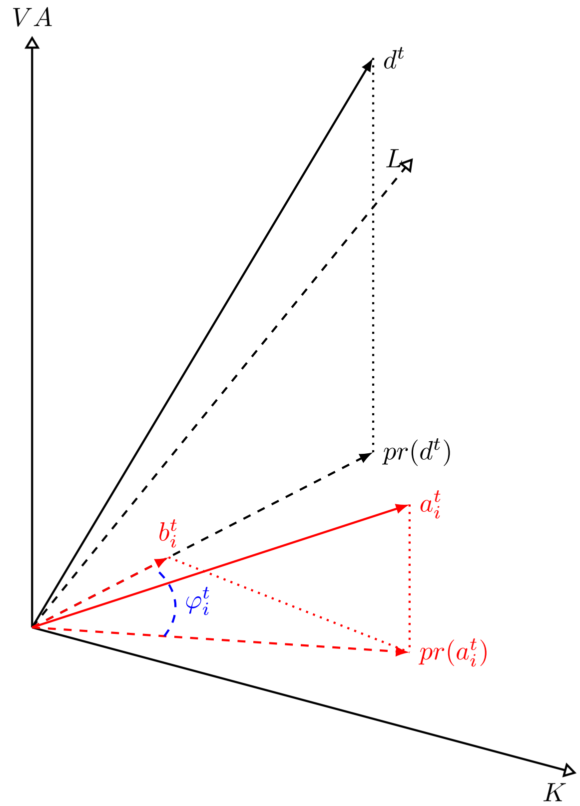
$$||pr\left(a_i^t\right)|| = \frac{||b_i^t||}{\cos\varphi_i^t} \quad ,$$

we can decompose w_i^t as

$$w_i^t = \frac{||pr\left(a_i^t\right)||}{||pr\left(d^t\right)||} = \frac{||b_i^t||}{\cos\varphi_i^t}\frac{1}{||pr\left(d^t\right)||} = \frac{||b_i^t||}{||pr\left(d^t\right)||}\frac{1}{\cos\varphi_i^t}$$

[5]All the mathematical details which lead to this decomposition can be found in appendix A.

Figure 1: A graphical explanation for the decomposition of the between effect



that is as the product:

$$w_i^t = s_i^t \cdot h_i^t \tag{13}$$

of what we could call the "**input weights**":

$$s_i^t = \frac{||b_i^t||}{||pr\left(d^t\right)||} \tag{14}$$

and a "**heterogeneity coefficient**":

$$h_i^t = \frac{1}{\cos \varphi_i^t} . \tag{15}$$

Equation (13) shows that the input based weights w_i^t in (11) combine together two different effects. The first one, s_i^t, represents the contribution of individual firms to the length of the industry input vector, i.e. our equivalent to the input weights in an multiple input case. Notice that in this case we have $\sum_{i \in C} s_i^t = 1$ and that, in the case of only one input s_i^t is the standard input share weight. The second one, that we named "heterogeneity coefficient" h_i^t, measures to which degree the individual input combinations diverge from the industry average combination. The larger this divergence, the bigger the angle φ_i^t and therefore the coefficient h_i^t. Thus, the sum $\sum_{i \in C} h_i^t$ can be regarded as an index of the heterogeneity of input combinations among productive units, and $\sum_{i \in C} \Delta h_i^t$ measures the variations of such heterogeneity.

Given these two effects, we can further decompose the *Between* term in (12) into two parts and thus refine our decomposition of APG as follows:[6]

$$\Delta P^t = \underbrace{\sum_{i \in C} \bar{w}_i \Delta p_i^t}_{Within} + \underbrace{\sum_{i \in C} \bar{p}_i \bar{h}_i \Delta s_i^t}_{InputShares} + \underbrace{\sum_{i \in C} \bar{p}_i \bar{s}_i \Delta h_i^t}_{Heterogeneity} \tag{16}$$

where the *Within* term is the same as the one in (12) and the sum of *InputShares* plus *Heterogeneity* is equal to the *Between* term in (12). The term *InputShares* measures the changes of contributions of individual firms to the industry total input size, whereas the term *Heterogeneity* measures the changes of heterogeneity among the combinations of inputs used by different firms. It is easy to see that when there is only one input, h_i^t is always equal to 1, *Heterogeneity* is 0 and the two decompositions (16) and (12) coincide.

As an illustration, consider the hypothetical example of Table 3 which describes an industry made of three firms. In year 1 such firms are identical, but in year 2 one of them has switched to a more capital intensive technique, moving from production activity vector $(1.814, 0.842, 1)$ to $(0.564, 1.919, 1)$ where output has remained unchanged and equal to 1 and the length of the input vector has also remained equal to 2.

Table 3 reports data for each firm and for the industry (last row). In each panel the last two columns report the productivity measures computed, respectively, with our methodology and with the standard one. The latter consists of TFPs for the individual firms (assuming production function (5)) and of the three weighted averages for the industry, where weights are given, respectively, by labour, capital and quantity shares.

Table 3: Example: Between, Input Shares and Heterogeneity Effects.

	Year 1					Year 2				
	L_i	K_i	Q_i	$tg(\cdot)$	TFP_i	L_i	K_i	Q_i	$tg(\cdot)$	TFP_i
Firm 1	1.814	0.842	1.000	0.500	0.668	1.814	0.842	1.000	0.500	0.668
Firm 2	1.814	0.842	1.000	0.500	0.668	1.814	0.842	1.000	0.500	0.668
Firm 3	1.814	0.842	1.000	0.500	0.668	0.564	1.919	1.000	0.500	1.305
	$\sum L_i$	$\sum K_i$	$\sum Q_i$	$tg(\cdot)$	w.a. TFP_i	$\sum L_i$	$\sum K_i$	$\sum Q_i$	$tg(\cdot)$	w.a. TFP_i
Industry	5.442	2.526	3.000	0.500	0.668 0.668 0.668	4.192	3.603	3.000	0.543	0.754 1.007 0.880

* w.a. stands for weighted average.

If we use our methodology we find that all individual productivities remain constant, whereas AP increases from 0.5 to 0.543 and this variation can be decomposed as follows:

$$\underbrace{0.043}_{APG} = \underbrace{0}_{Within} \underbrace{-0.001}_{InputShares} + \underbrace{0.044}_{Heterogeneity} \tag{17}$$

These values correctly reflect that individual productivities have remained unchanged (firm 3 has switched to a new technique which delivers the same output with a vector input of the same length and then has grown with constant returns to scale), so the *Within* effect is null, whereas *Heterogeneity* has considerably increased.

[6]Appendix A contains the mathematical details of the derivation of equation 16.

On the other hand, when we measure productivities with TFPs, we observe that the productivity of firms 3 has almost doubled (because of the different exponents of L and K in (5)) and therefore also AP undergoes an increment entirely due to the Within effect. For instance, if we use output shares as weights, APG is equal to 0.212 and can be decomposed as follows:

$$\underbrace{0.212}_{APG} = \underbrace{0.212}_{Within} + \underbrace{0}_{Between} \tag{18}$$

It may be worth explaining into greater detail the origin of such a substantial difference between the two decompositions. In the standard method a production function is imposed on data and, in our example, such a function reads the change of (L, K) combination of firm 3 as an increase of its productivity which, in turn, translates into a positive within effect in the aggregate. Our methodology instead interprets such a change as an increase of heterogeneity because firms 3 was using at time 1 the same input combination as the other two firms, whereas at time 2 its combination is substantially different. But since we do not impose a production function on data we cannot conclude that $(1.814, 0.842, 1)$ is less productive than $(0.564, 1.919, 1)$ as the two techniques have equal output and the same input size (i.e. the length of the vector of inputs). Nevertheless the change of technique of firm 3 determines a movement of the industry diagonal which results in an aggregate productivity increase. The Input Shares effect is negative because the diagonal moves towards firm 3 and therefore the projections of firms 1 and 2 on it become shorter.

3.4 Decomposing APG with Entry and Exit

In this section we expand our decomposition of APG by accounting for the contributions given by production units which enter or exit the industry during the period under consideration. Let C be the set of all continuing firms, i.e. those that are active both in $t-1$ and in t, N the set of entering firms which are active in t but not in $t-1$, and X the set of exiting firms which are active in $t-1$ but not in t. Let vector sets $\{a_i^{t-1}\}_{i\in\{C\cup X\}} \in \mathbb{R}_+^l$ and $\{a_i^t\}_{i\in\{C\cup N\}} \in \mathbb{R}_+^l$ represent all firms active in the industry in $t-1$ and t respectively. According to equation (10) we have

$$\begin{aligned}\Delta P^t &= \sum_{i\in\{C\cup N\}} w_i^t p_i^t - \sum_{i\in\{C\cup X\}} w_i^{t-1} p_i^{t-1} \\ &= \sum_{i\in C}(w_i^t p_i^t - w_i^{t-1} p_i^{t-1}) + (\sum_{i\in N} w_i^t p_i^t - \sum_{i\in X} w_i^{t-1} p_i^{t-1})\end{aligned}$$

where for all the continuing firms, the term $\sum_{i\in C}(w_i^t p_i^t - w_i^{t-1} p_i^{t-1})$ can be further decomposed as in equation (16) and finally we have

$$\begin{aligned}\Delta P^t = & \underbrace{\sum_{i\in C} \bar{w}_i \Delta p_i^t}_{Within} + \underbrace{\sum_{i\in C} \bar{p}_i \bar{h}_i \Delta s_i^t}_{InputShares} + \underbrace{\sum_{i\in C} \bar{p}_i \bar{s}_i \Delta h_i^t}_{Heterogeneity} \\ & + \underbrace{\sum_{i\in N} w_i^t p_i^t - \sum_{i\in X} w_i^{t-1} p_i^{t-1}}_{NetEntry} \quad .\end{aligned} \tag{19}$$

3.5 An Illustrative Example

Table 4 provides an illustrative example of a hypothetical industry composed of 5 production units (firms) producing a unique output with two inputs, L and K. The first three columns report the input-output data in year 1. The fourth and fifth columns report the length $||pr\,(a_i^1)\,||$ of the input vectors and the length $||a_i^1||$ of the production activity vectors respectively. In the sixth column, we compute productivities according to equation(8). Columns 7th to 12th report the same data referred to year 2. The only change taking place between the two years is an increase of the heterogeneity of the input combinations among firms, as visualized in Figure 2.[7] The industry becomes more productive, as productivity increases from 0.5001 to 0.5294. Applying decomposition method in equation (16) this 0.0293 increase of productivity can be decomposed in the following way:

$$\underbrace{0.0293}_{APG} = \underbrace{0}_{Within} \underbrace{-0.0006}_{InputShares} + \underbrace{0.0299}_{Heterogeneity} \quad .$$

which confirms that the within effect is null and the between effect is basically due to the increase of heterogeneity among firms, which is indeed the only phenomenon taking place between the two years.[8]

Table 4: Toy Example with Five Firms and Increasing Heterogeneity

	Year 1						Year 2					
	$1st$	$2nd$	$3rd$	$4th$	$5th$	$6th$	$7th$	$8th$	$9th$	$10th$	$11th$	$12th$
	L	K	Output	length of input vector	length of vector	$tg(\cdot)$	L	K	Output	length of input vector	length of vector	$tg(\cdot)$
Firm 1	1.414	1.414	1.000	2.000	2.236	0.500	1.414	1.414	1.000	2.000	2.236	0.500
Firm 2	1.464	1.362	1.000	2.000	2.236	0.500	1.764	0.942	1.000	2.000	2.236	0.500
Firm 3	1.424	1.404	1.000	2.000	2.236	0.500	1.864	0.724	1.000	2.000	2.236	0.500
Firm 4	1.374	1.453	1.000	2.000	2.236	0.500	1.044	1.706	1.000	2.000	2.236	0.500
Firm 5	1.394	1.434	1.000	2.000	2.236	0.500	0.564	1.919	1.000	2.000	2.236	0.500
Industry	7.071	7.068	5.000	9.998	11.178	0.5001	6.651	6.705	5.000	9.444	10.686	0.529

Let us now assume that in year 3 everything remains unchanged from year 2 except that firm 3 doubles its output with the same inputs. The first six columns of Table 5 report these hypothetical data for year 3. Because of the increase of firm 3's productivity, AP increases from 0.5294 to 0.6353 and this increase can be decomposed as:

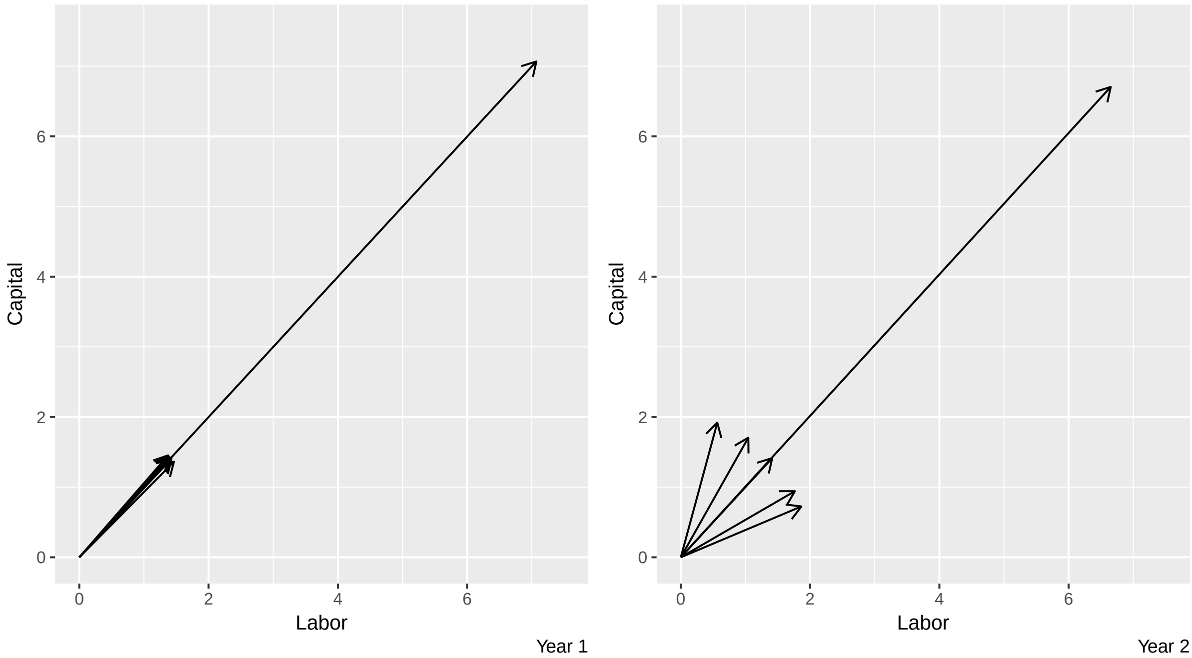
$$\underbrace{0.1059}_{APG} = \underbrace{0.1059}_{Within} + \underbrace{0}_{InputShares} + \underbrace{0}_{Heterogeneity}$$

which indicates that APG is completely driven by the technical change operated by Firm 3, and therefore only the $Within$ term is different from zero. Finally, let us suppose that between year 3 and year 4 all firms hold their productivities and heterogeneity coefficients h_i^t constant and only the input shares weights s_i^t change (the last six columns in Table 5). APG is now totally imputed to variations of the input weights, i.e. to the $InputShares$

[7] Vectors in Figure 2 are the projections $pr(a_i)$ on the space of inputs of the vectors a_i. So, for example, the projection of the three dimensional vector $a_1^1 = (1.414, 1.414, 1.000)$ of firm 1 in year 1 is simply given by the two dimensional vector $pr(a_1^1) = (1.414, 1.414)$.

[8] Consistently, the heterogeneity measure introduced by Dosi et al. (2016) increases from 2.09025e-06 to 0.00728504.

Figure 2: Toy Example with Five Firms and Increasing Heterogeneity



effect:

$$\underbrace{-0.0060}_{APG} = \underbrace{0}_{Within} \underbrace{-0.0060}_{InputShares} + \underbrace{0}_{Heterogeneity} .$$

These hypothetical examples show that our measure correctly captures the phenomena driving APG. The next section provides an empirical application to real data.

Table 5: Toy Example with Five Firms - Dynamic from Year 3 to Year 4

	Year 3							Year 4						
	L	K	Output	$\lVert pr(\cdot)\rVert$	$tg(\cdot)$	s_i^t	h_i^t	L	K	Output	$\lVert pr(\cdot)\rVert$	$tg(\cdot)$	s_i^t	h_i^t
Firm 1	1.414	1.414	1.000	2.000	0.500	0.212	1.000	1.494	1.494	1.057	2.113	0.500	0.224	1.000
Firm 2	1.764	0.942	1.000	2.000	0.500	0.202	1.046	1.766	0.943	1.001	2.002	0.500	0.203	1.046
Firm 3	1.864	0.724	2.000	2.000	1.000	0.193	1.095	1.777	0.690	1.906	1.906	1.000	0.184	1.095
Firm 4	1.044	1.706	1.000	2.000	0.500	0.206	1.028	1.081	1.766	1.035	2.071	0.500	0.213	1.028
Firm 5	0.564	1.919	1.000	2.000	0.500	0.186	1.137	0.533	1.811	0.944	1.888	0.500	0.176	1.137
Industry	6.651	6.705	6.000	9.444	0.635	1.000	1.000	6.651	6.705	5.943	9.444	0.629	1.000	1.000

4 An Empirical Application

4.1 Data and methodology

In order to show that our methodology is also empirically relevant, in this section we apply our APG decomposition to real firm-level data and compare it to the results that standard methods would produce on the same data. The database we use is the October 2015 release of **AMADEUS**, a commercial database provided by Bureau van Dijk and containing balance sheets and income statements for over 21 million European firms over the period 2004-2013. We focus on firms from several industries at the 4-digit NACE classification for three European countries, namely France, Italy and UK. These industries have been randomly selected from those with at least 20 firms (including continuous, exiting, and entering firms) during the time period under investigation for all the three countries. For reasons of space we report here only the results for seven selected industries, listed in Table 6. Results for other industries are available upon request.

Number of employees and fixed assets are chosen as proxies for the two inputs, labour and capital, and turnover as a proxy for output. In some cases we add a third input, proxied by material costs. All these values, except the number of employees, are measured in thousands Euros and expressed in 2010 prices using the appropriate deflator for the 4-digit industry and the country under consideration[9].

Table 6: List of Selected Industries

NACE	Name of Industry
2014	Manufacture of other organic basic chemicals
2120	Manufacture of pharmaceutical preparations
2593	Manufacture of wire products, chain and springs
2630	Manufacture of communication equipment
2712	Manufacture of electricity distribution and control apparatus
2813	Manufacture of other pumps and compressors
2920	Manufacture of bodies (coachwork) for motor vehicles; manufacture of trailers and semi-trailers

We compute APG and its decomposition for two time periods: between 2004 and 2007 and between 2010 and 2013. We compute four-years rather than yearly variations because productivity, however measured, tends to be very sticky over time, hence a longer period is more suited to investigate its variation. Moreover, we omit from consideration 2008 and 2009, the two years when the economic crisis hit more harshly the countries under consideration. The analysis of APG in these years is indeed interesting *per se*, but we want to make sure that our results are not driven by very abrupt variations in those years mostly due to plunging output and high exit rates.

Table 7 shows APG and its decomposition, all computed following our methodology[10] for the period 2004-07 in the left panel and 2010-13 in the right panel. In the left panel, column 1 shows the value of APG, columns 2-4 the contributions to APG given by, respectively, entering, exiting and continuing firms. The latter is further decomposed in a $Within$ and *Between* effect in column 5. Then the *Between* effect of column 5 is further decomposed in a $InputShares$ and a $Heterogenenity$ effects computed as in (16) and reported in column 6. Finally, in column 7 we report an heterogeneity coefficient coherent with our methodology and introduced by Dosi et al. (2016).[11] Columns 8 to 14 report the same results for APG between 2010 and 2013.

The table contains some interesting results. For instance, for what concerns sector "*2014*" from 2004 to 2007, we observe positive $Within$ and negative *Between* effects in

[9]Deflators for 4-digit industries are provided by *Eurostat* (`https://ec.europa.eu/eurostat/data/database`). In a few cases the 4-digit deflators for a specific industry are not available, hence more aggregate deflators, e.g. 3-digit or 2-digit deflators for that country, have been used.

[10]The software developed for computing AP, APG and its decompositions following the methodology developed in this article has been written in R and is available upon request.

[11]The coefficient is a Gini volume coefficient and is given by the ratio between the volume of the zonotope formed by the actual production activities and the volume of the zonotope of an industry with the same size but maximum heterogeneity. For more details see Dosi et al. (2016) p. 885.

all three countries. The former points to an increase in productivity at the firm level, whereas the latter signals that the reallocation of input shares had a negative effect on industry productivity. An opposite dynamics is observed for instance in industry "*2120*" in UK and Italy. As far as the between term is further decomposed, one can notice that although the variation in input shares tends to dominate over the heterogeneity term, the latter is far from being irrelevant and in a few instances even outweighs the other. Other things being equal, a variation in the adopted techniques by firms has an effect on AP.

4.2 Comparison with Standard Methods on Empirical Data

Finally, we compare our methodology with the more traditional one on the same set of data.

As a standard benchmark we adopt the method proposed by Grilliches and Regev (1995) and summarized in equation (3) above. In order to estimate TFPs of individual firms, which are then averaged to determine AP, we follow Levinsohn and Petrin (2003) and use material costs as the additional *proxy* variable necessary in their methodology.[12] Notice that by doing this, we lose all the firms from UK and some from France and Italy for which there is no indication of material costs in our original dataset.

Using employment shares as weights, we compute APGs from 2004 to 2007 for each country/industry and report them in column 1 of Table 8. Decomposition results, computed according to (3), can be found in columns 2 (Entry and Exit) and 3 (Within and Between). Similar APGs and their decompositions but using output instead of labour shares as weights can be found in column 4 to 6. In column 7 we report APG computed with our methodology, and in columns 8-10 its decomposition into the various components: column 8 gives the *Entry* and *Exit* effects, column 9 the *Within* and *Between* and column 10 splits the latter into *InputShares* and *Heterogeneity* effects. Finally, column 11 gives our measure of variation of industry heterogeneity, proxied by the Gini coefficient for the industry zonotope.

The right panel of Table 8 (columns 12-22) replicates the same indicators for the productivity variations between 2010 and 2013.

In commenting the results in Table 8, we start by reminding that the absolute measures of APG provided by our proposed method and by the more traditional one are the outcome of two very different approaches, hence they cannot be directly compared. The related rates of change (reported in brackets below APG) provide a better benchmark for comparison and reveal that both in terms of signs and order of magnitude of the variation, the two methods point to a common direction, of course with some noteworthy exceptions. Note however that on average - and as expected - the differences across the methods remain large. When we focus on the decomposition terms across the two methods, similar comments apply.

[12] We use **Stata** command **levpet** (Petrin et al., 2004) to estimate TFP with number of employees as *free* variable, fixed assets as *capital* variable, and material cost as *proxy* variable respectively. The dependent variable is total revenue instead value added.

Table 7: APG Decompositions for Selected Industries in France, Italy and UK, 2004-07 and 2010-13 - Our methodology

		From 2004 to 2007									From 2010 to 2013								
NACE	Ctry.	APG	Entry	Exit	Continue	*Within*	*Btw.*	*InputShares*	*Heterogeneity*	*Gini* Growth (%)	APG	Entry	-Exit	Continue	*Within*	*Btw.*	*InputShares*	*Heterogeneity*	*Gini* Growth (%)
		1	2	3	4	5		6		7	8	9	10	11	12		13		14
2014	FR	1.60	0.31	-0.34	1.63	2.57	-0.94	-3.27	2.34	42.24	-0.61	0.67	-0.20	-1.08	-0.26	-0.81	-0.81	0	-40.92
2014	UK	0.31	0.36	-0.00	-0.05	5.46	-5.51	-10.58	5.07	37.99	0.30	0.32	-0.17	0.15	1.28	-1.12	-1.13	0.01	-10.47
2014	IT	-0.00	0.40	-0.01	-0.40	0.49	-0.89	-0.89	0	98.76	1.97	0.49	-0.14	1.63	-0.46	2.09	2.09	0	-3.70
2120	FR	0.89	0.25	-0.47	1.11	0.01	1.1	1.25	-0.14	-7.88	-0.32	0.41	-0.47	-0.25	-0.04	-0.21	-0.09	-0.12	-7.69
2120	UK	0.47	0.36	-0.01	0.12	-0.23	0.35	0.36	-0.01	2.98	-1.15	0.90	-0.19	-1.86	0.19	-2.05	-2.1	0.05	75.09
2120	IT	-0.12	0.17	-0.24	-0.04	-0.49	0.44	0.44	0	3.37	-0.27	0.81	-0.37	-0.72	-0.48	-0.24	-0.2	-0.04	-10.22
2593	FR	1.25	0.71	-1.11	1.65	0.54	1.11	1.09	0.02	-8.46	4.54	4.89	-2.74	2.39	0.17	2.22	2.23	-0.01	-6.48
2593	UK	0.45	1.13	-0.44	-0.24	0.69	-0.93	-0.96	0.02	36.36	-0.51	0.52	-0.15	-0.87	2.15	-3.02	-4.64	1.62	-17.64
2593	IT	0.41	0.86	-0.07	-0.38	0.14	-0.52	-0.52	0	21.33	-0.00	0.57	-0.18	-0.39	-0.04	-0.36	-0.36	0	10.44
2630	FR	1.96	2.64	-3.30	2.62	-1.93	4.56	4.64	-0.08	165.30	-8.18	2.90	-6.47	-4.61	-0.49	-4.12	-4.13	0.01	-56.58
2630	UK	-1.07	1.40	-0.25	-2.21	0.66	-2.87	-7.54	4.67	-15.48	-0.47	0.21	-0.26	-0.42	-0.61	0.18	0.11	0.07	-30.54
2630	IT	-2.26	0.88	-0.47	-2.67	-2.03	-0.64	1.31	-1.94	53.77	-1.71	0.17	-0.76	-1.12	-33.76	32.65	38.31	-5.66	-29.79
2712	FR	0.10	3.00	-1.09	-1.81	-0.01	-1.79	-1.78	-0.01	-23.71	-1.87	1.74	-1.70	-1.91	-0.03	-1.88	-1.88	0	102.80
2712	UK	6.44	5.63	-0.19	1.00	-3.63	4.63	5.84	-1.21	279.46	-5.40	0.20	-5.16	-0.44	-1.54	1.1	1.09	0.01	-76.85
2712	IT	0.58	0.17	-0.01	0.42	0.42	0	0	0	-11.67	-0.09	0.41	-0.04	-0.45	-1.52	1.07	1.08	-0.01	-25.45
2813	FR	-0.70	0.37	-2.00	0.93	-0.01	0.95	0.95	0	34.75	0.59	1.81	-2.28	1.05	-0.44	1.49	1.49	-0.01	-28.72
2813	UK	-2.67	0.47	-0.03	-3.12	0.09	-3.21	-3.26	0.05	109.51	0.20	0.06	-0.03	0.17	-0.15	0.31	0.32	0	-24.87
2813	IT	0.57	1.35	-0.14	-0.63	0.52	-1.16	-1.16	0	56.87	-0.45	0.76	-0.55	-0.66	-1.17	0.51	1.32	-0.8	1.08
2920	FR	1.00	2.12	-2.56	1.44	1.04	0.4	0.87	-0.46	6.02	-0.17	2.30	-4.67	2.20	-0.04	2.23	2.24	-0.01	-6.80
2920	UK	0.16	0.95	-0.26	-0.52	0.47	-0.99	-0.99	0	14.39	1.42	0.40	-0.27	1.29	1.26	0.02	0.03	-0.01	-2.67
2920	IT	1.00	1.29	-1.02	0.72	1.49	-0.77	-1	0.23	7.49	0.11	0.85	-0.10	-0.64	0.46	-1.09	-1.08	-0.01	-44.96

Table 8: APG Decompositions for Selected Industries in France and Italy 2004-07 and 2010-13 - Our methodology vs. Griliches and Regev (1995)

		From 2004 to 2007											From 2010 to 2013										
Decomposition Method		Decomposition (3)						Proposed Method (19)					Decomposition (3)						Proposed Method (19)				
NACE	Ctry.	APG (% Growth)	*Entry* *-Exit*	*Within* *Btw*	APG (% Growth)	*Entry* *-Exit*	*Within* *Btw*	APG (% Growth)	*Entry* *-Exit*	*Within* *Btw*	*InpSh* *Heter*	*Gini Growth* (%)	APG (% Growth)	*Entry* *-Exit*	*Within* *Btw*	APG (% Growth)	*Entry* *-Exit*	*Within* *Btw*	APG (% Growth)	*Entry* *-Exit*	*Within* *Btw*	*InpSh* *Heter*	*Gini Growth* (%)
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
2014	FR	1.77 (12.12)	1.05 -2.13	1.8 1.05	2.39 (14.28)	1.05 -2.73	4.59 -0.53	0.04 (2.53)	0.13 -0.27	0.43 -0.26	-0.4 0.14	-9.82	-3.35 (-20.44)	4.37 -1.5	-1.37 -4.85	-12.05 (-40.55)	2.56 -0.69	-5.07 -8.84	-0.18 (-11.59)	0.36 -0.1	-0.1 -0.34	-0.32 -0.02	-58.54
2014	IT	0.67 (25.82)	1.1 -0.03	0.43 -0.84	0.6 (23.94)	1.04 -0.03	0.34 -0.76	0.05 (4)	0.32 -0.01	0.17 -0.44	-0.5 0.06	291.02	-0.25 (-5.89)	1.75 -1.22	-0.49 -0.28	0.02 (0.81)	0.91 -0.7	-0.02 -0.15	0.39 (35.02)	0.22 -0.11	0.02 0.27	0.44 -0.17	-25.29
2120	FR	0.59 (2.95)	3.08 -3.38	0.61 0.27	3.25 (11.74)	1.41 -3.29	0.87 4.27	0.27 (13.21)	0.17 -0.35	-0.16 0.61	0.63 -0.02	-10.34	2.01 (10.15)	3.9 -3.13	0.53 0.7	-3.42 (-11.77)	3.55 -8.53	1.05 0.52	-0.07 (-3.84)	0.22 -0.27	-0.02 -0.01	-0.03 0.03	-22.48
2120	IT	0.52 (2.82)	1.34 -2.84	0.47 1.54	1.12 (5.89)	1.02 -2.16	0.96 1.3	0.02 (1.35)	0.1 -0.21	0.07 0.08	0.1 -0.03	15.61	-0.24 (-1.13)	7.74 -0.87	-0.05 -7.06	0.17 (0.8)	6.59 -2.8	0.31 -3.92	0.05 (2.78)	0.58 -0.21	-0.08 -0.24	-0.26 0.01	-10.96
2593	FR	-0.23 (-1.36)	2.53 -3.99	-0.13 1.36	-0.09 (-0.49)	1.89 -4.05	0.22 1.86	-0.07 (-3.94)	0.21 -0.4	0.02 0.11	0.17 -0.06	-26.31	-1.21 (-7.19)	8.75 -12.06	-0.26 2.36	-0.73 (-4.19)	9.46 -12.19	-0.1 2.1	-0.15 (-8.16)	0.97 -1.28	-0.02 0.19	0.2 -0.01	-35.31
2593	IT	0.05 (5.27)	0.36 -0.03	-0.05 -0.23	0.06 (7.56)	0.32 -0.03	-0.04 -0.19	-0.01 (-0.36)	0.45 -0.04	-0.08 -0.33	-0.34 0	10.30	0.25 (24.73)	0.54 -0.07	-0.04 -0.18	0.16 (17.93)	0.38 -0.06	-0.03 -0.12	0.06 (4.02)	0.4 -0.12	-0.02 -0.2	-0.21 0.01	47.68
2630	FR	1.65 (8.9)	3.81 -8.2	0.06 5.98	3.51 (18.65)	4.59 -7.79	1.02 5.69	-0.36 (-13.9)	0.53 -1.26	-0.25 0.61	0.7 -0.09	519.39	5.33 (28.04)	16.92 -11.04	-0.12 -0.44	5.68 (26.75)	21.54 -10.19	-0.54 -5.14	-0.23 (-8.6)	1.9 -1.53	-0.04 -0.57	-0.67 0.11	-73.20
2630	IT	-1.25 (-3.02)	9.62 -5.66	-0.17 -5.05	5.84 (11.11)	15.99 -5.95	1.58 -5.78	0 (-0.09)	0.5 -0.3	0.04 -0.25	-0.24 -0.01	104.81	3.11 (6.93)	3.96 -9.13	-44.3 52.57	1.05 (1.94)	5.09 -14.08	-39.54 49.57	-0.38 (-19.77)	0.17 -0.42	-28.14 28.02	33.91 -5.89	-14.13
2712	FR	0.4 (3.22)	5.99 -3.02	0.18 -2.76	0.49 (3.93)	7.13 -2.68	0.26 -4.22	-0.07 (-3.67)	1.04 -0.4	-0.06 -0.66	-0.67 0.02	-45.69	0.25 (1.88)	5.15 -3.68	0.11 -1.33	0.07 (0.53)	4.13 -3.15	0.12 -1.03	0.1 (5.21)	0.62 -0.44	0.01 -0.09	-0.09 0	22.45
2712	IT	0.48 (3.12)	2.41 -0.18	0.43 -2.19	0.53 (3.5)	1.36 -0.14	0.37 -1.06	0.35 (33.72)	0.14 -0.01	0.2 0.02	0.05 -0.03	24.67	-1.08 (-6.67)	3.93 -0.62	-1.59 -2.8	-3.92 (-21.16)	3.25 -0.34	-2.43 -4.4	-0.09 (-6.01)	0.32 -0.03	-0.14 -0.24	-0.18 -0.07	-55.92
2813	FR	1.52 (204.58)	0.05 -0.24	1.34 0.37	1.6 (172.92)	0.03 -0.23	1.97 -0.17	-0.19 (-8.58)	0.13 -0.69	-0.14 0.52	0.51 0.01	39.92	0.66 (46.69)	0.15 -0.48	0.65 0.34	1.88 (108.15)	0.14 -0.3	1.17 0.87	0.13 (6.3)	0.88 -1.21	-0.04 0.49	0.47 0.02	-46.77
2813	IT	1.38 (5.52)	7.68 -0.74	0.3 -5.86	2.17 (7.98)	9.13 -0.89	0.68 -6.76	0.02 (1.07)	0.51 -0.06	-0.01 -0.42	-0.41 -0.01	92.80	0.08 (0.3)	7.79 -3.18	-0.86 -3.67	-1.89 (-6.36)	7.97 -5.81	-1.54 -2.5	0 (0.17)	0.42 -0.27	-0.12 -0.03	-0.06 0.03	11.53
2920	FR	10.9 (17.17)	11.93 -12.61	7.29 4.29	14.08 (19.93)	11.2 -14.68	9.64 7.91	0.02 (1.25)	0.34 -0.44	0 0.13	0.14 -0.02	24.57	3.73 (6.51)	12.52 -24.6	1.53 14.28	-2.04 (-3.05)	14.53 -34.12	1.5 16.04	0.02 (1.09)	0.46 -0.92	0.01 0.46	0.47 0	-2.34
2920	IT	-4.83 (-7.28)	14.62 -11.4	-2.03 -6.02	-25.87 (-26.43)	14.82 -22.53	-14.98 -3.17	0.04 (2.75)	0.36 -0.33	-0.05 0.07	0.07 0	19.42	6.2 (14.99)	14.95 -10.94	1.16 1.03	-3.02 (-5.07)	13.28 -1.27	-0.64 -14.39	0.08 (5.44)	0.41 -0.05	0.04 -0.32	-0.32 0	-45.42

5 Conclusions

Thanks to the increasing availability of longitudinal establishment- and firm- level data, a rapidly growing body of empirical literature has shown a highly significant and persistent degree of heterogeneity among firms and establishments in the input combinations and in their productivities even in the presence of the same relative input prices and in narrowly defined industries, thus with relatively homogeneous types of output. Moreover, the analyses addressed the issue of the relative importance between firm-level increase in productivity and the reallocation of market share to the APG, i.e. so-called "within" and "between" effects, across individual producers within narrowly defined sectors. The message, overall, is that "within" effects, i.e. learning, dominate upon "between" effects, i.e. selection.

However, such widespread and persistent heterogeneity poses serious challenges to the use of standard aggregate production functions and AP one can derive from them. In addition, it challenges also the innocence of aggregations based on sheer weights given by input or output shares. In this article, building upon a geometric representation of the empirical production possibility set first suggest by Hildenbrand (1981) and developed in Dosi et al. (2016), we introduce a new decomposition method for APG which 1) computes individual and aggregate productivities in the same way, instead of computing the latter as some arbitrary weighted average of the individual indicators; 2) reduces the loss of information implied by standard decomposition methods; 3) allows for a precise measure of the contribution given by variations in heterogeneity.

Our methodology can be applied to empirical data and the preliminary application we present in this article, on some selected industries in France, Italy and the UK, show that indeed the contributions to APG that can be attributes to changes of firm-level heterogeneity are far from negligible.

To put it differently, given the overwhelming evidence on heterogeneity at all levels of observation, aggregation of "micro production functions" is not appropriate. However, it is also partly misleading to use simple "evolutionary decompositions" just relying on learning vs. selection components, as revealed by the often "perverse" sign of their interaction in empirical estimates (firms with higher productivity growth shrinking in size). Rather, it seems more informative to make explicit a heterogeneity component which is orthogonal to either, somehow similar to the drift component found in evolutionary biology.

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A Derivation of APG decomposition

In this Appendix we spell out the mathematical details behind the derivation of equations (12) and (16).

Decomposition in equation (12) By the decomposition of productivity as $P^t = \sum_{i \in C} w_i^t p_i^t$ described in equation (10) we get that

$$\Delta P^t = P^t - P^{t-1} = \sum_{i \in C} w_i^t p_i^t - \sum_{i \in C} w_i^{t-1} p_i^{t-1} \quad .$$

The above equality holds if we add and subtract same quantity from its right side, that is

$$\Delta P^t = \sum_{i \in C} w_i^t p_i^t - \sum_{i \in C} w_i^{t-1} p_i^{t-1} + \sum_{i \in C} w_i^t p_i^{t-1} - \sum_{i \in C} w_i^t p_i^{t-1} + \sum_{i \in C} w_i^{t-1} p_i^t - \sum_{i \in C} w_i^{t-1} p_i^t$$

which we can collect as

$$\Delta P^t = \sum_{i \in C} \frac{w_i^{t-1} + w_i^t}{2}(p_i^t - p_i^{t-1}) + \sum_{i \in C} \frac{p_i^{t-1} + p_i^t}{2}(w_i^t - w_i^{t-1})$$

which becomes equation (12)

$$\Delta P^t = \underbrace{\sum_{i \in C} \bar{w}_i \Delta p_i^t}_{Within} + \underbrace{\sum_{i \in C} \bar{p}_i \Delta w_i^t}_{Between}$$

using notation $\bar{w}_i = \frac{w_i^{t-1} + w_i^t}{2}, \bar{p}_i = \frac{p_i^{t-1} + p_i^t}{2}$ and $\Delta p_i^t = p_i^t - p_i^{t-1}, \Delta w_i^t = w_i^t - w_i^{t-1}$.

Decomposition in equation (16) Following decomposition

$$\Delta P^t = \sum_{i \in C} \bar{w}_i \Delta p_i^t + \sum_{i \in C} \bar{p}_i \Delta w_i^t$$

in equation (12), we further decompose the coefficient $\Delta w_i^t = w_i^t - w_i^{t-1}$ as follows. because $w_i^t = s_i^t h_i^t$ (see equation (13)), we get equality

$$\Delta w_i^t = s_i^t h_i^t - s_i^{t-1} h_i^{t-1}$$

which can be modified adding and subtracting same quantity in the right side as follows

$$\Delta w_i^t = s_i^t h_i^t - s_i^{t-1} h_i^{t-1} + (s_i^t h_i^{t-1} - s_i^t h_i^{t-1}) + (s_i^{t-1} h_i^t - s_i^{t-1} h_i^t) \quad .$$

Terms in the second part of the equality can be collected as

$$\Delta w_i^t = \frac{h_i^{t-1} + h_i^t}{2}(s_i^t - s_i^{t-1}) + \frac{s_i^{t-1} + s_i^t}{2}(h_i^t - h_i^{t-1}) \quad .$$

If we denote, as usual, by $\bar{h}_i = \frac{h_i^{t-1} + h_i^t}{2}$ and $\bar{s}_i = \frac{s_i^{t-1} + s_i^t}{2}$ the average sums, and by $\Delta s_i^t = s_i^t - s_i^{t-1}$ and $\Delta h_i^t = h_i^t - h_i^{t-1}$ the variations, then Δw_i^t becomes

$$\Delta w_i^t = \bar{h}_i \Delta s_i^t + \bar{s}_i \Delta h_i^t$$

which, replaced in equation (12), gives

$$\Delta P^t = \underbrace{\sum_{i \in C} \bar{w}_i \Delta p_i^t}_{Within} + \underbrace{\sum_{i \in C} \bar{p}_i \bar{h}_i \Delta s_i^t}_{InputShares} + \underbrace{\sum_{i \in C} \bar{p}_i \bar{s}_i \Delta h_i^t}_{Heterogeneity}$$

that is equation (16).

B APG decomposition with multiple outputs

In this appendix we give a quick description of the AP and its decomposition when firms produce more than one output.

Notations and definitions During period t the production unit i, which is described by the vector

$$a_i^t = \left(\alpha_{i,1}^t, \cdots, \alpha_{i,m}^t, \alpha_{i,m+1}^t, \cdots \alpha_{i,m+n}^t\right) \in \mathbb{R}_+^{m+n} \quad , \tag{20}$$

produces $a_{i,out}^t = \left(\alpha_{i,m+1}^t, \cdots \alpha_{i,m+n}^t\right)$ units of n kinds of outputs by means of $a_{i,in}^t = \left(\alpha_{i,1}^t, \cdots, \alpha_{i,m}^t\right)$ units of m kinds of inputs. Denote by I^t the set of all firms within one industry at time t. Then the set of vectors $\{a_i^t\}_{i\in I^t} \in \mathbb{R}_+^{m+n}$ represents the production activities of firms within the same industry at time t. Thus the aggregate (industry) production activity d^t is the sum of individual firm production activity and can be written as

$$\begin{aligned} d^t &= \left(\beta_1^t, \cdots, \beta_m^t, \beta_{m+1}^t, \cdots, \beta_{m+n}^t\right) \\ &= \left(\sum_{i\in I^t} \alpha_{i,1}^t, \cdots, \sum_{i\in I^t} \alpha_{i,m}^t, \sum_{i\in I^t} \alpha_{i,m+1}^t \cdots, \sum_{i\in I^t} \alpha_{i,m+n}^t\right) \in \mathbb{R}_+^{m+n} . \end{aligned} \tag{21}$$

If we denote by

$$\begin{aligned} pr_{out} : \mathbb{R}^{m+n} &\to \mathbb{R}^n \\ a_i^t &\mapsto a_{i,out}^t \end{aligned}$$

and by

$$\begin{aligned} pr_{in} : \mathbb{R}^{m+n} &\to \mathbb{R}^n \\ a_i^t &\mapsto a_{i,in}^t \end{aligned}$$

the projections of the production activities a_i^t (analogously of the industry vector d^t) on the spaces of outputs and inputs respectively, then the formula to compute the industry and firms productivities become

$$P^t := tg\left(\theta(d^t)\right) = \frac{||pr_{out}\left(d^t\right)||}{||pr_{in}\left(d^t\right)||} \tag{22}$$

and

$$p_i^t := tg\left(\theta(a_i^t)\right) = \frac{||pr_{out}\left(a_i^t\right)||}{||pr_{in}\left(a_i^t\right)||} \tag{23}$$

respectively, where $\theta(.)$ denotes the angle of vectors d^t and a_i^t with the space of inputs. Notice that, because in this case output $a_{i,out}^t$ is a multidimensional vector then, in general, $||pr_{out}\left(d^t\right)|| \neq \sum_{i\in I^t} ||pr_{out}\left(a_i^t\right)||$, unless all output vectors are proportional or there is only one output. If we denote by φ_i^t the angle formed by the vectors $pr_{in}\left(a_i^t\right)$ and $pr_{in}\left(d^t\right)$ and by σ_i^t the angle formed by the vectors $pr_{out}\left(a_i^t\right)$ and $pr_{out}\left(d^t\right)$, we get that

$$||pr_{out}\left(d^t\right)|| = \sum_{i\in I^t} \left(||pr_{out}\left(a_i^t\right)|| \cos\sigma_i^t\right) \tag{24}$$

which replaced into equation (22) gives

$$P^t = \frac{\sum_{i\in I^t}\left(||pr_{out}\left(a_i^t\right)||\cos\sigma_i^t\right)}{||pr_{in}\left(d^t\right)||} = \sum_{i\in I^t}\left(\cos\sigma_i^t\frac{||pr_{in}\left(a_i^t\right)||}{||pr_{in}\left(d^t\right)||}\frac{||pr_{out}\left(a_i^t\right)||}{||pr_{in}\left(a_i^t\right)||}\right) ,$$

that is

$$P^t = \sum_{i\in I^t} u_i^t p_i^t$$

where the "weight" coefficient

$$u_i^t := k_i^t \cdot w_i^t$$

is defined as the product of the output-based-homogeneity measure

$$k_i^t := \cos\sigma_i^t$$

and the input-based-weight

$$w_i^t := \frac{||pr_{in}\left(a_i^t\right)||}{||pr_{in}\left(d^t\right)||} .$$

Notice that k_i^t is a decreasing function of σ_i^t when $\sigma_i^t \in \left[0, \frac{\pi}{2}\right]$. That is smaller σ_i^t's correspond to bigger k_i^t's and indicate that the vector $pr_{out}(a_i^t)$ is closer to the vector $pr_{out}(d^t)$, i.e. less output-based-heterogeneity. The fact that more output-based-homogeneity coincides with bigger k_i^t explains why we name k_i^t as output-based-homogeneity measure.

Decomposing the aggregate industry growth Let's denote by C, N, and X the sets of continues, entering, and exiting firms respectively. APG from time $t-1$ to time t is given by

$$\Delta P^t = \sum_{i\in C}\bar{u}_i\Delta p_i^t + \sum_{i\in C}\bar{p}_i\Delta u_i^t + \sum_{i\in N}u_i^t p_i^t - \sum_{i\in X}u_i^{t-1}p_i^{t-1} . \tag{25}$$

where for any variable $x^t \in \mathbb{R}$ at time t, operator Δ represents its change from $t-1$ to t, i.e. $\Delta x^t \equiv x^t - x^{t-1}$, and $\bar{x} \equiv \frac{x^t + x^{t-1}}{2}$. We can further decompose Δu_i^t as

$$\begin{aligned}\Delta u_i^t &= \frac{k_i^{t-1}+k_i^t}{2}(w_i^t - w_i^{t-1}) + \frac{w_i^{t-1}+w_i^t}{2}(k_i^t - k_i^{t-1})\\ &= \bar{k}_i\Delta w_i^t + \bar{w}_i\Delta k_i^t\\ &= \bar{k}_i\left(\bar{h}_i\Delta s_i^t + \bar{s}_i\Delta h_i^t\right) + \bar{w}_i\Delta k_i^t \end{aligned} \tag{26}$$

where the last step substitutes the decomposition of $\Delta w_i^t = w_i^t - w_i^{t-1}$ as indicated in appendix A. Finally, by substituting (26) into (25) we have

$$\Delta P^t = \underbrace{\sum_{i\in C}\bar{u}_i\Delta p_i^t}_{Within} + \underbrace{\sum_{i\in C}\bar{p}_i\bar{k}_i\bar{h}_i\Delta s_i^t}_{InputShares} + \underbrace{\sum_{i\in C}\bar{p}_i\bar{k}_i\bar{s}_i\Delta h_i^t}_{Heterogeneity} + \underbrace{\sum_{i\in C}\bar{p}_i\bar{w}_i\Delta k_i^t}_{Homo^{out}} + \underbrace{\sum_{i\in N}u_i^t p_i^t - \sum_{i\in X}u_i^{t-1}p_i^{t-1}}_{NetEntry} . \tag{27}$$

Notice that when $n = 1$, i.e. there is only one output, the angle σ_i^t between the individual output vector and the aggregate output vector degenerates to 0. Thus for all firm i over all time t, we have

$$k_i^t = \cos\sigma_i^t = 1$$

and thus $\bar{k}_i = 1$, $\Delta k_i^t = 0$ and $u_i^t = w_i^t$. As a result, the decomposition (27) degenerates to (19).