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Giulio Bottazzi

Advanced Calculus for Economics and Finance

Theory and Methods

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To Cilli and Gigi

Preface

This document started as a collection of notes prepared for the courses of advanced calculus for undergraduate and graduate students of economics held at the Scuola Sant'Anna. It has been a work in progress for more than ten years. There are multiple purposes of this work. First, the treatment of some topics, most notably topological and metric spaces, but also measure theory, is developed in a rather axiomatic way, starting from general definitions and then analysing what these definitions imply under different conditions. The axiomatic approach is an excellent tool for training students in logical deduction, logical consistency, and formalisation of ideas. Attention is paid here not only to the final results, but also to how those results are derived and what type of logical relation connects them. For this purpose, I decided to clearly separate the analysis of topological, normed, and metric spaces, which are often merged in introductory and middle-level calculus textbooks. Second, while presenting material which is already part of basic introductory courses on calculus, such as numerical sequences, series, and differential calculus, I follow a theorem-proving approach, stressing the boundary of application of the different theorems, often attempting to provide thought-provoking counterexamples. I also include notable results that are often used in applied mathematics books, but that I have found to be constantly missing in basic- and mid-level mathematical courses. For many topics, this book provides far more results than the usual book on calculus, even if not the kind of coverage one could find in a specialised publication. The idea is to teach the reader the appropriate language and notions about each topic so that they can understand where and how to look for more specific discussions should the need arise. The main criterion that drove the selection of topics is usability in applications, in particular, applications in economics and social sciences. This is why I have avoided almost entirely any geometric consideration, which is important in a course for hard scientists and engineers, and I have insisted on mathematical results mostly useful for optimisation theory and statistics. The two final appendices, on the initial value problem for systems of differential equations and on the Brower fixed-point theorem, exemplify two possible domains of application of the material covered in this book.

I have several people to thank. First, the Allievi of Scuola Sant'Anna, who, at different levels, have followed my courses. They asked the questions that this document was designed to answer and gave me invaluable feedback on the text itself.

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Giulio Bottazzi

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