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Equilibrium Return and Agents' Survival in a Multiperiod Asset Market: Analytic Support of a Simulation Model

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Summary. We provide explanations for the results of the Levy, Levy and Solomon model, a recent simulation model of financial markets. These explanations are based upon mathematical analysis of a dynamic model of a market with an arbitrary number of heterogeneous investors allocating their wealth between two assets. The investors' choices are endogenously modeled in a general way and, in particular, consistent with the maximization of an expected utility. We characterize the equilibria of the model and their stability and discuss implications for the market return and agents' survival. These implications are in agreement with the results of previous simulations. Thus, our analytic approach allows to explore the robustness of the previous analysis and to expand its spectrum.

19.1 Introduction

The goal of this paper is to explore analytically the framework underlying simulations of the so-called Levy, Levy and Solomon (henceforth LLS) model. The model was introduced in [6] and further results were presented in [7] and [9], among others. See also [8] for extensive discussion. The motivation behind the model was to investigate whether some financial anomalies (like excess volatility or autocorrelation of returns) can be explained by relaxing the traditional assumption of classical finance about the presence of a fully-informed and rational representative agent. The LLS framework assumes the presence of heterogeneous agents whose market impact depends on their past performances. In the words of its authors ([8], p. 143):

> "The LLS model incorporates some of the main empirical findings regarding investor behavior, and we employ this model to study the effect of each element of investor behavior on asset pricing and market dynamics."

The model has been shown to qualitatively explain many of the financial anomalies, but all its results are based on simulations. The criticism of the

simulation approach usually points at a huge number of degrees of freedom, i.e. dimensions of a set consisting of (i) all possible parameters, (ii) realizations of the random variables and (iii) initial conditions. This leads to the feeling that "everything one wants to obtain" can be obtained in the heterogeneous world. In other words, the absence of a closed form solution makes it difficult to believe that the results are robust. As a reply to that criticism, many analytic models of financial markets with heterogeneous agents appeared, see [5] for a recent review. On the other hand, an analytic approach is limited due to the high non-linearity of the models with heterogeneous agents. For example, the agents' wealth evolution is usually neglected in analytic contributions. Therefore, both analytic and simulation approaches have to co-exist and to supplement each other. As we show with our analysis, the analytic investigations of the LLS model can effectively supplement the results of previous simulation exercises.

Our analytic model of the LLS framework starts off with a pure exchange, two-assets economy, where agents invest according to different rules. The framework is consistent with the CRRA (Constant Relative Risk Aversion) behavior, so that the individual demand for the risky asset is expressed as a fraction of the agent's wealth. Consequently, the price and agents' wealths are determined simultaneously, and, moreover, agents with different wealth levels have different impact on the price realization.

Models in [2, 3, 4] are predecessors of our model. In particular, as in [3], we model the agents' behavior by means of generic *investment functions*, mapping the available information on the current investment choice. However, we substantially deviate from these papers since we introduce a more realistic dividend process. Instead of assuming a constant dividend yield, we analyse the case where the dividend is growing at a given constant rate. This system corresponds to the *deterministic skeleton* of a market where dividend follows a geometric random walk. We provide equilibrium and stability analysis for this skeleton, which sheds light on the behavior of the stochastic LLS model, where the growth rate of dividends is random.

The direct application of our analytic model follows from the fact that the market structure we use is the same as in simulations of the LLS model. In [6, 7, 8, 9] the agents are expected utility maximizers having power utility function. One of the obstacles on the way to explore such setting analytically is the absence of a closed-form solution for the corresponding optimization problem. This obstacle has played a role in arguments in favor of simulations. However, in our framework with investment functions the precise solutions are not necessary, since the analytic results are expressed in terms of the general functions and can be illustrated geometrically. The difficulty of dealing with a power utility function is overcome, and comparative statics exercises can be easily performed, analogously to what has been done in [1]. Thus, our analysis allows to explain simulation results that alternatively have to be described in an rather vague fashion as in the following quote from [9] (p.568, 569):

> "Looking more systematically at the interplay of risk aversion and memory span, it seems to us that the former is the more relevant factor, as with different [risk aversion coefficients] we frequently found a reversal in the dominance pattern: groups which were fading away before became dominant when we reduced their degree of risk aversion. [...] It also appears that when adding different degrees of risk aversion, the differences of time horizons are not decisive any more, provided the time horizon is not too short."

The rest of the paper is organized as follows. In Section 19.2 the analytic model is presented. In Section 19.3 the main results of the equilibrium and stability analysis are summarized in a few propositions. In Section 19.4 we apply these results and, therefore, offer a rigorous explanation of the findings in [7] and [9], among others. The analytic results also help to discuss the robustness of the simulation results with respect to the different assumptions. We also present some further results in order to characterize the dynamics when the equilibria are unstable. Section 19.5 concludes.

19.2 Model Structure

Let us consider N agents trading in discrete time in a two-asset economy with a riskless asset giving a constant interest rate $r_f > 0$ and constant supply (normalized to 1) of risky asset paying a random dividend D_t. The price of the riskless asset is fixed to 1, and the price P_t of the risky asset is fixed through market clearing. Let $W_{t,n}$ stand for the wealth of agent n at time t and $x_{t,n}$ for its share invested in the risky asset. The dividend is paid before trade starts, so the wealth evolves as

$$W_{t+1,n} = (1 - x_{t,n})\, W_{t,n}\, (1 + r_f) + \frac{x_{t,n}\, W_{t,n}}{P_t}\, (P_{t+1} + D_{t+1})\,. \tag{19.1}$$

The price at time t is fixed through the market clearing condition

$$\sum_{n=1}^{N} x_{t,n}\, W_{t,n} = P_t\,. \tag{19.2}$$

Assume that the agent's investment share $x_{t,n}$ does not depend upon the wealth. The resulting demand is consistent with the one derived from the maximization of a constant relative risk aversion (CRRA) utility function. Moreover the investment shares are independent of the contemporaneous price and bounded between zero and one, $x_{t,n} \in (0, 1)$, for all t and n. Both assumptions are consistent with previous simulations of the LLS model and simplify the analysis substantially. Notice that according to (19.1), the wealth does depend upon the contemporaneous price, so that price and wealth are simultaneously determined by the market clearing condition (19.2). Thus, (19.1) and (19.2) give the evolution of the state variables $W_{t,n}$ and P_t over time implicitly, provided that the investment shares $\{x_{t,n}\}$ are specified.

Concerning the latter we further assume that for each agent n there exists an *investment function* f_n such that

$$x_{t,n} = f_n(\mathcal{I}_t), \tag{19.3}$$

where $\mathcal{I}_t = \{D_t, D_{t-1}, \dots, P_{t-1}, P_{t-2}, \dots\}$ is the information set available to the agents at time t. Agents' investment decisions evolve following individual prescriptions. The generality of the investment functions allows a big flexibility in the modeling of the agents' behaviors. Formulation (19.3) includes as special cases both technical trading, when agents' decisions are driven by the observed price fluctuations, and more fundamental attitudes, e.g. when the decisions are made on the basis of the price-dividend ratio. It also includes the case of constant investment strategy, occurring when agent assumes the stationarity of the *ex-ante* return distribution.

For our application in Section 19.4 it is important to stress that (19.3) includes those investment behaviors which are derived from expected utility maximization with power utility $U(W, \gamma) = W^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ is the relative risk aversion coefficient. Indeed, solution of such a problem has a wealth independent investment share. This property holds for any distribution of the next period return which the agent is assumed to perceive now and for any risk aversion. However, the solution is unavailable in explicit form. Consequently, the analysis of the LLS model in [6, 7, 8, 9] rely on numeric solutions. Since the results of Section 19.3 are valid for any given functional form f, provided some easy-to-check general properties, we are able to perform an analytic analysis of the LLS model even when agents maximize expected utility with power utility function.

Accordingly with the LLS model, assume that $D_t = D_{t-1}(1+\tilde{g})$, where the growth rate, $\tilde{g}$, is an i.i.d. random variable whose mean is g. Below we perform an analysis of the *deterministic skeleton* of the dynamics triggered by this stochastic process, and we fix the growth rate of dividends to a constant value g.

With some algebra one can show that the implicit dynamics described in (19.1) and (19.2) can be made explicit. The resulting system is written in terms of the price return $k_{t+1} = P_{t+1}/P_t - 1$, dividend yield $y_{t+1} = D_{t+1}/P_t$ and agents' relative wealth shares in the aggregate wealth $\varphi_{t,n} = W_{t,n}/\sum_m W_{t,m}$ as follows

$$\begin{cases} y_{t+1} = y_t \dfrac{1+g}{1+k_t} \\ k_{t+1} = r_f + \dfrac{\sum_m \left((1+r_f)(x_{t+1,m} - x_{t,m}) + y_{t+1}\, x_{t,m}\, x_{t+1,m}\right)\varphi_{t,m}}{\sum_m x_{t,m}(1 - x_{t+1,m})\varphi_{t,m}} \\ \varphi_{t+1,n} = \varphi_{t,n} \dfrac{(1+r_f) + (k_{t+1} + y_{t+1} - r_f)\, x_{t,n}}{(1+r_f) + (k_{t+1} + y_{t+1} - r_f) \sum_m x_{t,m}\varphi_{t,m}} \\ x_{t+1,n} = f_n\left(k_t, k_{t-1}, \dots, k_{t-L}; y_{t+1}, y_t, \dots, y_{t-L}\right) \end{cases} . \tag{19.4}$$

The numerator of the fraction in the right-hand side of the third equation in (19.4) represents the wealth return of agent n. Thus, the relative wealth changes in accordance with the agent's performance relative to the average performance, where the return of individual wealth should be taken as a performance measure. The second equation in (19.4) stresses the role of the agents' relative wealths in the return determination: the richer agents have higher influence on the market. Finally, the last equation in (19.4) specifies the information set $\mathcal{I}_t$ in terms of the same variables as other equations. For the further analysis we assume that agents base their behavior on the finite number of past price returns and dividend yields. Their memory span L can be arbitrarily large, however.

19.3 Equilibrium Return and Agents' Survival

Given the arbitrariness of the size of population N and absence of any specification for the investment function, the analysis of the dynamic behavior generated by system (19.4) is highly non-trivial in its general formulation. One may, indeed, expect that nothing specific can be said about the dynamics. Let us, however, limit ourselves to the "equilibrium" situations, corresponding to the fixed points of system (19.4). In this Section we investigate how such equilibria can be characterized, under which conditions they represent the long-run behavior of the system (in other words, when they are stable), and which agents have positive wealth shares, i.e. survive, in the equilibria. The proofs of all statements are available upon request.

19.3.1 Location of Equilibria

The following result allows us to classify all possible equilibria into two classes, depending upon the values of two exogenous variables.

Proposition 1. *Let us consider the equilibrium of the system (19.4) given by the dividend yield y^*, return k^*, investment shares $(x_1^*, \ldots, x_n^*)$ and wealth distribution $(\varphi_1^*, \ldots, \varphi_n^*)$.*

The two following cases are possible:

(i). $g > r_f$. Then $k^* = g$, and all survivors (agents with non-zero wealth shares) have the same investment share $x_\diamond^*$, which together with y^* satisfies

$$\frac{g - r_f}{y^*} = \frac{x_\diamond^*}{1 - x_\diamond^*}. \tag{19.5}$$

(ii). $g \le r_f$. Then $k^* = r_f$ and $y^* = 0$.

In both cases the wealth shares of survivors are arbitrary positive numbers summing to 1, while the agent's investment shares satisfy $x_n^* = f_n(k^*, \ldots, k^*; y^*, \ldots, y^*)$, with corresponding k^* and y^*.

This result shows that the equilibrium price return is $k^* = \max(g, r_f)$. If the dividend growth rate is smaller than r_f, the dividend yield converges to zero, and the risky asset asymptotically yields the same return as the riskless asset. In this case, the equilibria described in Proposition 1*(ii)* are referred as **no-equity premium equilibria** (NEPE). The investment shares of agents are unambiguously determined through the investment functions, while the wealth shares are free of choice, so any number of agents can survive in such equilibria. Notice that NEPE imply zero dividend yield and, therefore, are unfeasible, strictly speaking. They can be observed asymptotically, however.

If the dividend grows fast enough, so that $g > r_f$, the equilibrium dividend yield y^* depends on agents' behaviors. From (19.5) one can easily show that the risk premium in such an equilibrium is positive and equal to $(g - r_f)/x^*_\diamond$. Consequently, the equilibria from Proposition 1*(i)* are called the **equity premium equilibria** (EPE). Even if the EPE can have any number $M \in \{1, \ldots, N\}$ of survivors, all of them must behave identically and invest $x^*_\diamond$. This is the key result for getting a simple geometric characterization of the EPE. Indeed, it implies that all possible couples "dividend yield – survivor's investment share" belong to a one-dimensional curve, which is introduced below.

Definition 1. *The Equilibrium Market Line (EML) is the following function*

$$l(y) = \frac{g - r_f}{y + g - r_f} \qquad \textit{defined for} \quad y > 0. \tag{19.6}$$

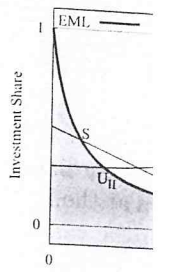
Now it follows from (19.5), that the dividend yield in the EPE with M survivors (which are the first M agents, without loss of generality) should satisfy to M equations

$$l(y^*) = f_n(g, \ldots, g; y^*, \ldots, y^*) \qquad \forall n \in \{1, \ldots, M\}.$$

In other words, the dividend yield in the EPE can be found as an intersection of the EML with M one-dimensional functions representing the "diagonal" cross-sections of the original investment functions by the set

$$\{k_t = k_{t-1} = \cdots = k_{t-L} = g; \quad y_{t+1} = y_t = \cdots = y_{t-L} = y\}. \tag{19.7}$$

The left panel of Fig. 19.1 illustrates the EPE in the market with two different agents, whose investment functions (more precisely, diagonal cross-sections of the original investment functions) are shown as thin lines marked as I and II. Their three intersections with the EML, shown as a thick line, give all the possible EPE. At equilibrium S the agent I is the only survivor, so that $\varphi^*_1 = 1$. The dividend yield y^* at this equilibrium is the abscissa of the point S, while the investment share of the survivor, x^*_1, is the ordinate of S. Finally, the investment share of the second agent can be found as a value of his investment function at y^*. Notice that in this equilibrium $x^*_1 > x^*_2$. Analogously, the variables are determined in other two equilibria. In particular, agent I is the



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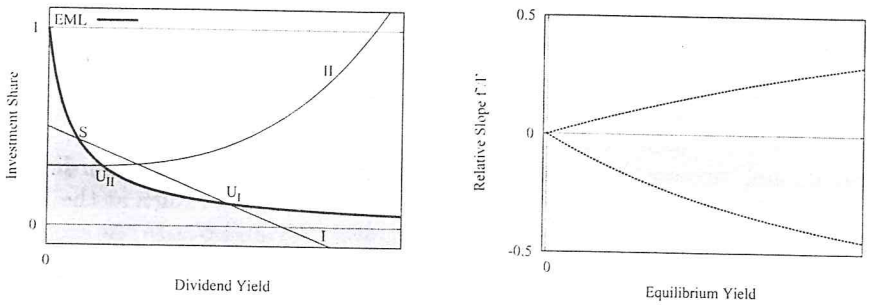

Fig. 19.1. Location and stability of equilibria for $g > r_f$. **Left Panel:** EPE are intersections of the EML with the investment functions; **Right Panel:** Stability of equilibria for $L = 1$.

only survivor at equilibrium U_I, while at U_{II} the second agent survives, $\varphi_2^* = 1$.

In all equilibria illustrated in Fig. 19.1 only one agent survives. In the case of more then one survivors, Proposition 1*(i)* implies that their investment functions should have a common intersection with the EML. Such situation is rather special, while the illustrated example can be classified as "generic".

Finally, with some simple algebra, one can characterize the agents' wealth growth rates in different equilibria.

Corollary 1. *(i). The wealth return of agent n is equal to* $1 + r_f + x_n^*(g - r_f)/x_\diamond^*$ *at any EPE. Thus, the wealth of all survivors grows at the same rate* g.

(ii). At the NEPE the wealth of all the agents grows at the same rate r_f.

19.3.2 Stability of Equilibria

The next natural question concerns the stability of the equilibria characterized in Proposition 1. In this paper we investigate this question only for the case $g > r_f$, i.e. only for the equity premium equilibria. The following general result holds.

Proposition 2. *The EPE, described in Proposition 1(i). where the first* M *agents survive, is stable if and only if the following conditions are met:*

1) the equilibrium investment shares of the non-surviving agents are such that

$$x_\diamond^* \left(1 - 2(1+g)/(g - r_f)\right) < x_m^* < x_\diamond^* \qquad \forall m \in \{M+1, \ldots, N\}. \tag{19.8}$$

2) after eliminating all non-surviving agents, the behavior of survivors generates stable dynamics.

This Proposition gives an important necessary condition for stability of the EPE. Namely, investment shares of non-surviving agents must satisfy (19.8). The leftmost inequality is always satisfied for reasonable values of g and r_f, while the rightmost inequality shows that the survivors should behave more aggressively in equilibrium, i.e. invest higher investment share, than those who do not survive. This result is intuitively clear, because, according to Corollary 1, the most aggressive agent has a higher wealth return at the EPE. Proposition 2 implies the instability of equilibria U_I and U_{II} in the example shown on the left panel of Fig. 19.1. In the stable equilibrium the investment shares of non-surviving agents should belong to the gray area.

If condition (19.8) is satisfied, the non-survivors can be eliminated from the market. When is the resulting equilibrium stable? We answer this question only for the case of single survivor with investment function dependent upon the average of past L total returns

$$x_t = f\Big(\sum\nolimits_{\tau=1}^{L} (y_{t-\tau} + k_{t-\tau})/L\Big)\,. \tag{19.9}$$

This special case will be important in the applications of Section 19.4. Standard stability analysis leads to the following result.

Proposition 3. *Let (x^*, y^*, k^*) be an EPE with one survival agent. The EPE is asymptotically stable if and only if all the roots of polynomial*

$$Q(\mu) = \mu^{L+1} - \frac{1+\mu+\cdots+\mu^{L-1}}{L}\Big(1 + (1-\mu)\frac{1+g}{y}\Big)\,\frac{f'(y^*+g)}{l'(y^*)} \tag{19.10}$$

lie inside the unit circle.

From Section 19.3.1 it follows that the equilibrium yield at the EPE is given as a solution of $l(y) = f(y+g)$. Thus, the last fraction in the polynomial (19.10) gives the relative slope of the investment function and the EML at the equilibrium. On the EML plot, this is the relative slope of the cross-section of an investment function and the EML in the intersection.

Propositions 2 and 3, give exhaustive characteristics of stability conditions of the EPE with single survivor in the market where agents behave according to (19.9). The stability conditions are implicit, however, since they contain a requirement on the roots of polynomial $Q(\mu)$. When $L = 1$ this requirement can be made explicit. Namely, the following two inequalities are sufficient for stability:

$$\frac{f'(y^*+g)}{l'(y^*)} > \frac{-y^*}{1+g+y^*} \qquad \text{and} \qquad \frac{f'(y^*+g)}{l'(y^*)} < \frac{y^*}{y^*+2(1+g)}\,.$$

These conditions are illustrated in the right panel of Fig. 19.1 in the coordinates $(y^*, f'/l')$. The equilibrium is stable if it belongs to the gray area.

A mixture of analytic and numeric tools helps to reveal the behavior of the roots of polynomial (19.10) with higher L, and, therefore, to understand

the impact of the agent's memory span on the stability of corresponding equilibrium. In general, the equilibrium stabilizes with lower (in absolute value) relative slope f'/l' at the equilibrium and with higher memory span L.

19.4 Analytic Support of Simulations

All the simulations of the LLS model deal with agents who maximize a power utility function with relative risk aversion γ, and who use the average of the last L returns as an estimate for the next period return. Even if the investment function for such an optimization problem cannot be derived explicitly, one can investigate how the cross-section of this function by the hyperplane (19.7) changes with parameters γ and L. In this Section we show that this is sufficient for explaining the results of the simulations in [6, 7, 8, 9]. We start the analysis by illustrating the effects of the risk aversion and memory span in the case of mean-variance investment function (which can be derived explicitly). The insights developed in this case will then be used to discuss the results of the original simulations. Throughout this Section it is assumed that $g > r_f$, so that only EPE are analyzed.

Let us consider an agent who maximizes the following mean-variance utility

$$U = E_t[x_t(k_{t+1} + y_{t+1}) + (1 - x_t)r_f] - \frac{\gamma}{2} V_t[x_t(k_{t+1} + y_{t+1})], \qquad (19.11)$$

where E_t and V_t denote, respectively, the mean and the variance conditional on the information available at time t, and γ is the coefficient of risk aversion. Assuming constant expected variance $V_t = \sigma^2$, the optimal investment fraction is $x_t^* = E_t[k_{t+1} + y_{t+1} - r_f]/(\gamma\sigma^2)$. Consistently with the LLS framework, we assume that the next period return is estimated as the average of past L realized return, while the expected variance is constant, and we bound the investment shares in the interval $[0.01, 0.99]$. Thus, the investment function reads

$$f_{\alpha,L} = \min\left\{0.99, \max\left\{0.01, \frac{1}{\alpha}\left(\frac{1}{L}\sum\nolimits_{\tau=1}^{L}(k_{t-\tau} + y_{t-\tau}) - r_f\right)\right\}\right\},$$

where we have defined $\alpha = \gamma\sigma^2$. From Section 19.3.1 it follows that all the EPE can be found as the intersections of the EML with the function

$$\tilde{f}_\alpha(y) = \min\left\{0.99, \max\left\{0.01, \frac{y + g - r_f}{\alpha}\right\}\right\},$$

which is the cross-section of $f_{\alpha,L}$ by the hyperplane (19.7). The left panel of Fig. 19.2 illustrates the situation for a single agent. The market has a unique equilibrium, A_α, whose abscissa, y_α^*, is the equilibrium dividend yield, and whose ordinate, x_α^*, is the equilibrium agent's investment share. This

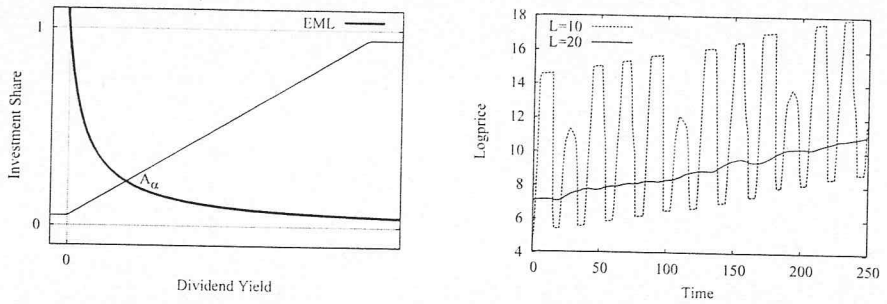

Fig. 19.2. Market dynamics with a single mean-variance maximizer. **Left panel:** The equilibrium on the EML. **Right panel:** Log-price dynamics over the simulations for two different values of memory span L. In both cases the investment function is the one depicted in the left panel.

equilibrium does not depend on the memory span L, but depends on the (normalized) risk aversion coefficient α. When α increases, the line $x = (y + g - r_f)/\alpha$ rotates counter-clock wise. Therefore, the equilibrium yield is an increasing function of the risk aversion, while the equilibrium investment share is a decreasing function of the risk aversion.

What are the determinants of the stability of the equilibrium A_α? First of all, notice that the stability analysis of Section 19.3.2 can be applied, because the investment function $f_{\alpha,L}$ is of the type specified in (19.9). The stability, therefore, is determined both by the relative slope of the function $\tilde{f}_\alpha$ with respect to the EML in point A_α and by the memory span L. In particular, the increase of L brings stability to the system.

The right panel of Fig. 19.2 shows the log-price time series resulting from two simulations of the model for the investment function $\tilde{f}_\alpha$ in the case where the dividend follows a geometric random walk. The only difference between simulations lies in the memory span L. The dotted line shows dynamics for the agent with $L = 10$. The equilibrium is unstable in this case, and the endogenous fluctuations which we observe are determined by the upper and lower bounds of $\tilde{f}_\alpha$. Moreover, the period of fluctuations is related to L. The solid line shows the price series obtained with memory increased to $L = 20$. The system converges to the stable equilibrium, and the fluctuations are due to exogenous noise affecting the dividend growth rate. Notice that a different α value may require a different minimum value of L to produce a stable equilibrium.

We now turn to the analysis of a market with many agents. In this case we are particularly interested in assessing the agents' survival. For this purpose we use the results of Proposition 2, namely that the survivor should have the highest investment share at his intersection with the EML. If, being alone, the survivor generates a stable equilibrium, he also dominates the market, i.e. asymptotically has all the wealth. The top left panel of Fig. 19.3 shows

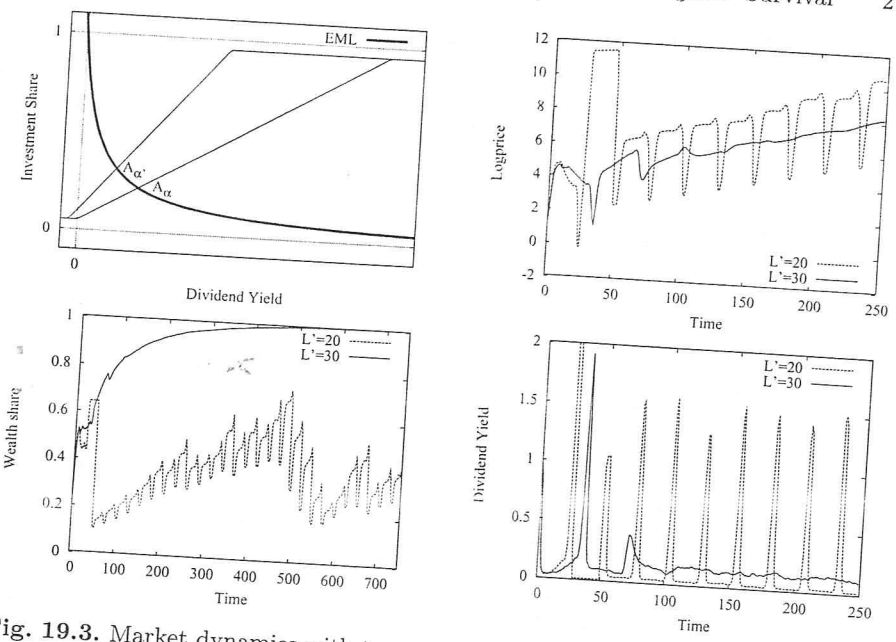

Fig. 19.3. Market dynamics with two mean-variance maximizers. **Top left panel:** EML and two investment functions. The agent with lower risk aversion α' produces equilibrium A'. **Top right panel:** Log-price dynamics. **Bottom left panel:** Dynamics of the wealth share of the agent with lower risk aversion α'. **Bottom right panel:** The dividend yield dynamics. All simulations are performed for two levels of memory span of the agent with low risk aversion.

two investment functions for different values of risk aversion, namely α and $\alpha' < \alpha$. Since at y^*_α the agent with risk aversion α invests less then the other agent, he cannot survive at "his" equilibrium, A_α, and, therefore, he can never dominate the market. Whether the agent with risk aversion α' is able to dominate the market depends on the stability of "his" equilibrium, $A_{\alpha'}$. If the memory he uses is long enough, the equilibrium is stable and $\varphi^*_{\alpha'} = 1$.

Fig. 19.3 shows the results of simulations for two different values of the memory parameter L' for the agent with lower risk aversion α'. When $L' = 20$, this agent, while destroying the previously stable equilibrium A_α does not bring the system to a new equilibrium. In fact, he destabilizes the price dynamics but fails to dominate the market and his wealth share keeps fluctuating between zero and one. However, when the memory of this agent increases to $L' = 30$, the new equilibrium $A_{\alpha'}$ is stabilized and he ultimately dominates the market. The equilibrium return now converges to $g + y^*_{\alpha'} < g + y^*_\alpha$. Thus, the agents with a lower risk aversion dominates the market, but produce lower equilibrium yield by investing a higher wealth share in the risky asset.

This analysis helps to explain results of the simulations in [7] and [9], and their findings concerning the interplay between risk aversion and memory. We have seen that the risk aversion is mostly related to the capability of agents to invade the market, whereas the memory span influences the stability of the dynamics. These properties hold as long as the investment function on the "EML plot" shifts upward with decrease of the risk aversion. It is easy to see that the investment function, coming from expected utility maximization with power utility has the same general features as mean-variance function used in the examples above. In fact, for a given y and a given perceived variance σ^2, the agents with lower risk aversion invest more, which guarantees the upward shift of the cross-section. As a result, Propositions 3 and 2 can be used. They provide rigorous analytic support of the simulation results of the LLS model.

In [7] the focus is on the role of the memory. The authors show that with a small memory span the log-price dynamics is characterized by crashes and booms. Our analysis shows that this is due to the presence of an unstable equilibrium and to the upper and lower bounds of the investment shares. Furthermore, this equilibrium becomes stable if the memory is high enough. Simulations in [7] confirm this statement; when agents with higher memory are introduced, booms and crashes disappear and price fluctuations become erratic. But as we found, these fluctuations are due to the exogenous noise (coming from the dividend) and not to the endogenous agents' interactions.

In [9] the focus is on the interplay between the length of the memory span and risk aversion. The simulations suggest that the risk aversion is more important than memory in the determination of the dominating agents, providing that the memory is not too short (see the quote in Section 19.1). Our analytic results explains why this is the case. Namely, it is because agents with low risk aversion are able to destabilize the market populated by agents with high risk aversion. However, this "invasion" leads to an ultimate domination only if the invading agents have sufficiently long memory. Otherwise, agents with different risk aversion coefficients will coexist. Notice that this result is new compared to [9] and related works. Thus, our analytic investigation is indeed helpful in understanding the interplay between risk aversion and memory. Another new result concerns the case of agents investing constant fraction of wealth. In [9] the authors claim that such agents always dominate the market and add (p. 571):

> "Hence, the survival of such strategies in real-life markets remains a puzzle within the Levy, Levy and Solomon microscopic simulation framework as it does within the Efficient Market Theory."

Our analysis allows one to understand and also correct this statement. The agents with constant investment fraction are characterized by the horizontal investment functions, for which Proposition 3 guarantees stability independently of L. If these agents are able to invade the market successfully, they will ultimately dominate. However, they cannot invade the market when other agents invest more in their EPE.

Finally, notice that for the case $g > r_f$, which we discuss here, Corollary 1 implies that the economy grows with rate g. All our present and all previous simulations are in accord with this statement. The case $g < r_f$ appears in [6], where the dividend is constant, so that $g = 0$, while the risk-free rate is positive. The resulting price grows with rate r_f, as we can expect from Corollary 1.

19.5 Conclusion

We have performed an analytic investigation of the LLS model and used its results to explain simulations in [6, 7, 8, 9]. We show that the two parameters governing the profitability of the risky and riskless investment opportunities, dividend growth rate and risk-free interest, and determine whether the equity premium can be endogenously generated at equilibrium. The size of equity premium depends on the agents' behavior. We have shown how the stability of the equilibria is related to the memory span that agents use to estimate future returns and their risk aversion. The results are very general and can help understand and extend the findings of previous simulations even when the functional form of the investment function is not known explicitly.

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